Projectors, Reflections, Rotations

We have already learned about reflections on a mirror plane spanned by two *Cartesian* unit vectors and rotations about one of the *Cartesian* axes. But what about reflections on arbitrary planes and rotations about arbitrary axes?

- Define the longitudinal and transverse projection operators P_{L} and P_{T} as well as the transverse rotation operator R_{T} (cross product matrix). Find their multiplication table.
- How could you tell if a given matrix describes a rotation, a reflection, an inversion or a combination of any of them?
- Find the matrix for reflections on arbitrary planes (containing the origin) with a normal unit vector $\boldsymbol{n}$. For a given reflection matrix, find the corresponding plane.

- Find the matrix of a rotation about an arbitrary axis (crossing the origin). For a given rotation matrix, find the corresponding rotation axis and angle.

- **Bonus:** Perform a decomposition of the rotation by $2\pi/3$ about the space diagonal into a product of fundamental rotations R_{z-x-z} and R_{z-y-x}. Why is the first decomposition usually preferred, even if it does not rotate at least once about each of the three Cartesian axes?

Notes

Because of the various ambiguities wrt. rotation matrices, we agree on the following conventions:

Alibi (active) transformation
We rotate points/vectors, not the coordinate system (alias/passive transformation).

Pre-multiplication of column vectors
We use the operator form, i.e. $R\boldsymbol{x}$, where a column vector is Pre-multiplied by the matrix in contrast to post-multiplication, where a row vector is multiplied with a matrix, i.e. $\boldsymbol{x}R$.

Handedness
We use solely right-handed coordinate systems, no left-handed ones. Consequently, we may use the "right hand rule" to determine the sign of rotation angles.

Extrinsic rotations
During a composition of rotations about different angles, the coordinate system stays fixed. It is also possible to rotate it with an object, which is called intrinsic rotation.

Euler angles
For decomposing an arbitrary rotation matrix, you may use the following reference matrix (z-x-z convention)[1, §19.4], [2-4]

$$R_{z-x-z} = R_z(\gamma)R_x(\beta)R_z(\alpha) = \begin{pmatrix} c_1c_3 - s_1c_2s_3 & -s_1c_3 - c_1c_2s_3 & s_2s_3 \\ c_1s_3 + s_1c_2c_3 & -s_1s_3 + c_1c_2c_3 & -s_2c_3 \\ s_1s_2 & c_1s_2 & c_2 \end{pmatrix}, \tag{1}$$

where $s_{1/2/3}$ and $c_{1/2/3}$ are shortcuts for sine and cosine of the Euler angles α, β, γ (not Tait-Bryan angles!). Further, we agree on the ranges $\alpha \in [0, 2\pi), \beta \in [0, \pi]$ and $\gamma \in [0, 2\pi)$ in order to find unique sets (at least in case $x_3' \neq x_3$). For a decomposition in rotations about the three Cartesian axes, you may use

$$R_{z-y-x} = R_z(\gamma)R_y(\beta)R_x(\alpha) = \begin{pmatrix} c_2c_3 & s_1s_2c_3 - c_1s_3 & c_1s_2c_3 + s_1s_3 \\ c_2s_3 & s_1s_2s_3 + c_1c_3 & c_1s_2s_3 - s_1c_3 \\ -s_2 & s_1c_2 & c_1c_2 \end{pmatrix}. \tag{2}$$

References

[1] Tilo Arens, ed. *Mathematik.* 2., korrigierter Nachdr. OCLC: 633452574. Heidelberg: Spektrum, Akad. Verl, 2010. ISBN: 978-3-8274-1758-9.

See also the following Wikipedia entries and included graphics / animations

[2] https://en.wikipedia.org/wiki/Rotation_matrix#Ambiguities

[3] https://en.wikipedia.org/wiki/Rotation_formalisms_in_three_dimensions

[4] https://en.wikipedia.org/wiki/Euler_angles

These two handouts give a good overview of the topic wrt. linear algebra

[5] robotics.caltech.edu/~jwb/courses/ME115/handouts/rotation.pdf

[6] http://scipp.ucsc.edu/~haber/ph216/rotation_12.pdf