



Airborne Wind Energy

Basic mechanics of crosswind kites II

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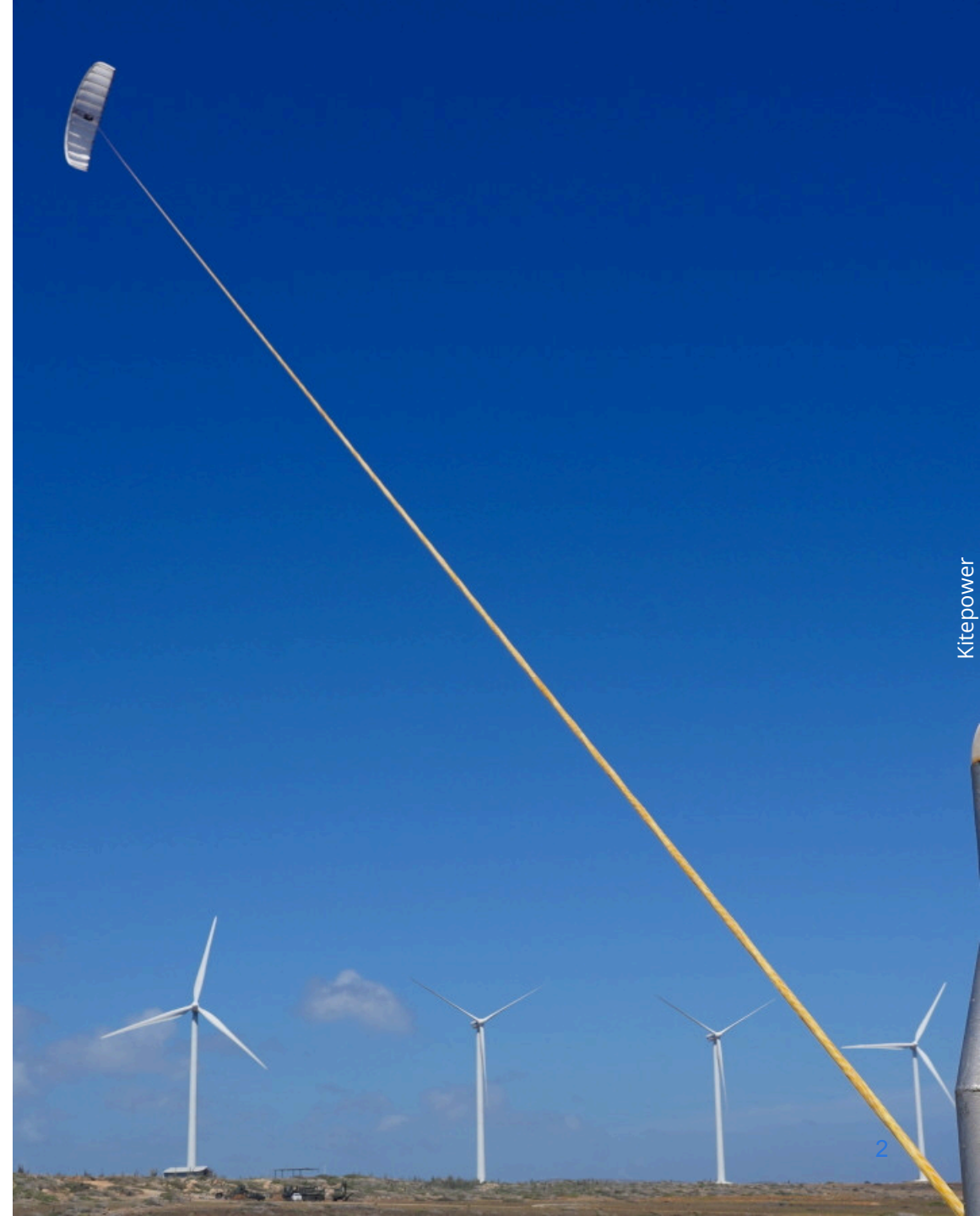
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Outlines

- Ideal crosswind flight
- Crosswind kite with mass
- Dynamic take-off wind speed
- Radial flight





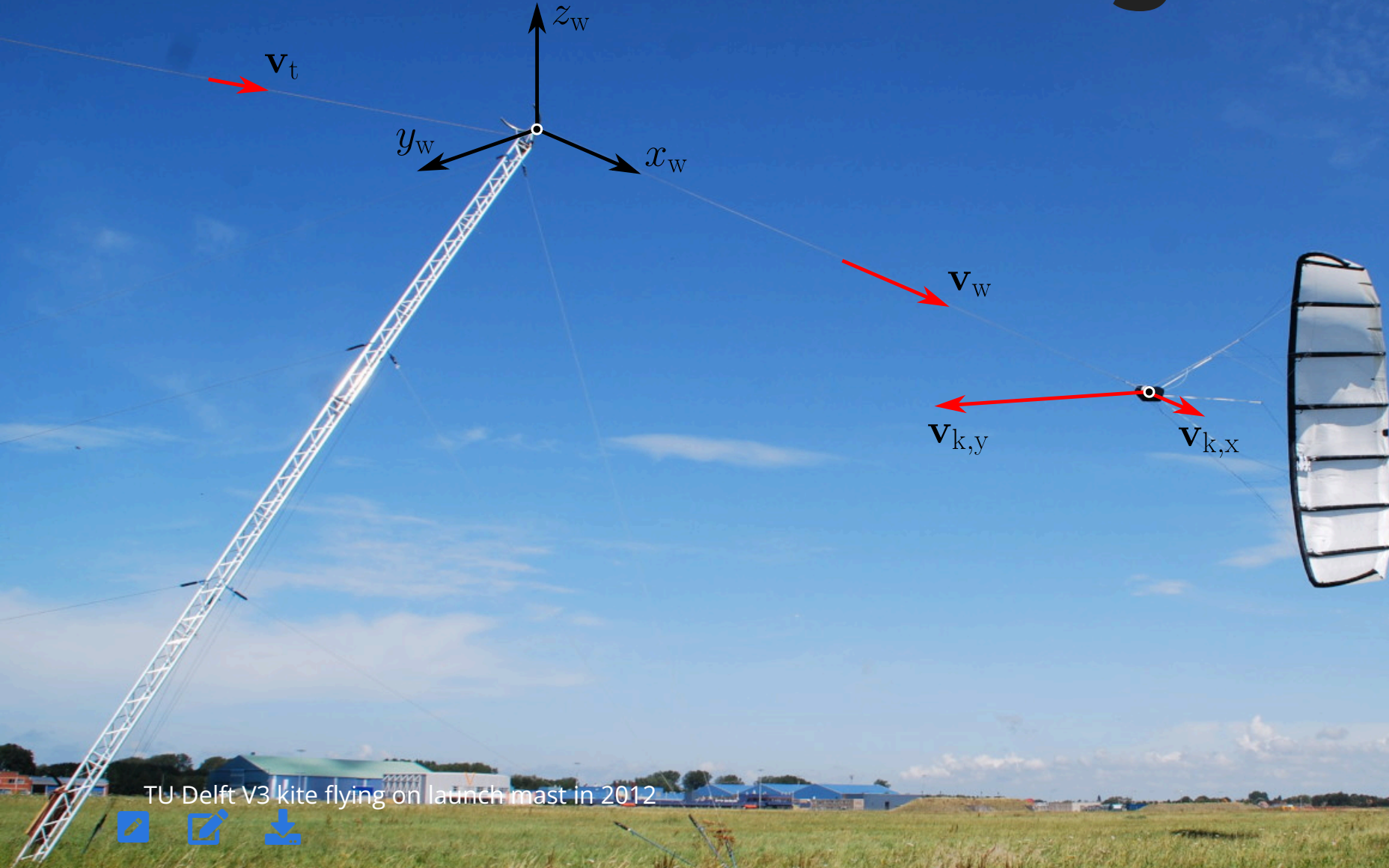
Learning objectives

- Analyze Loyd's ideal crosswind kite
- Derive the dynamic take-off wind speed
- Analyze Loyd's case of radial flight

Relevant literature sources are Loyd ([1980](#)) and Diehl ([2013](#)).



Ideal crosswind flight

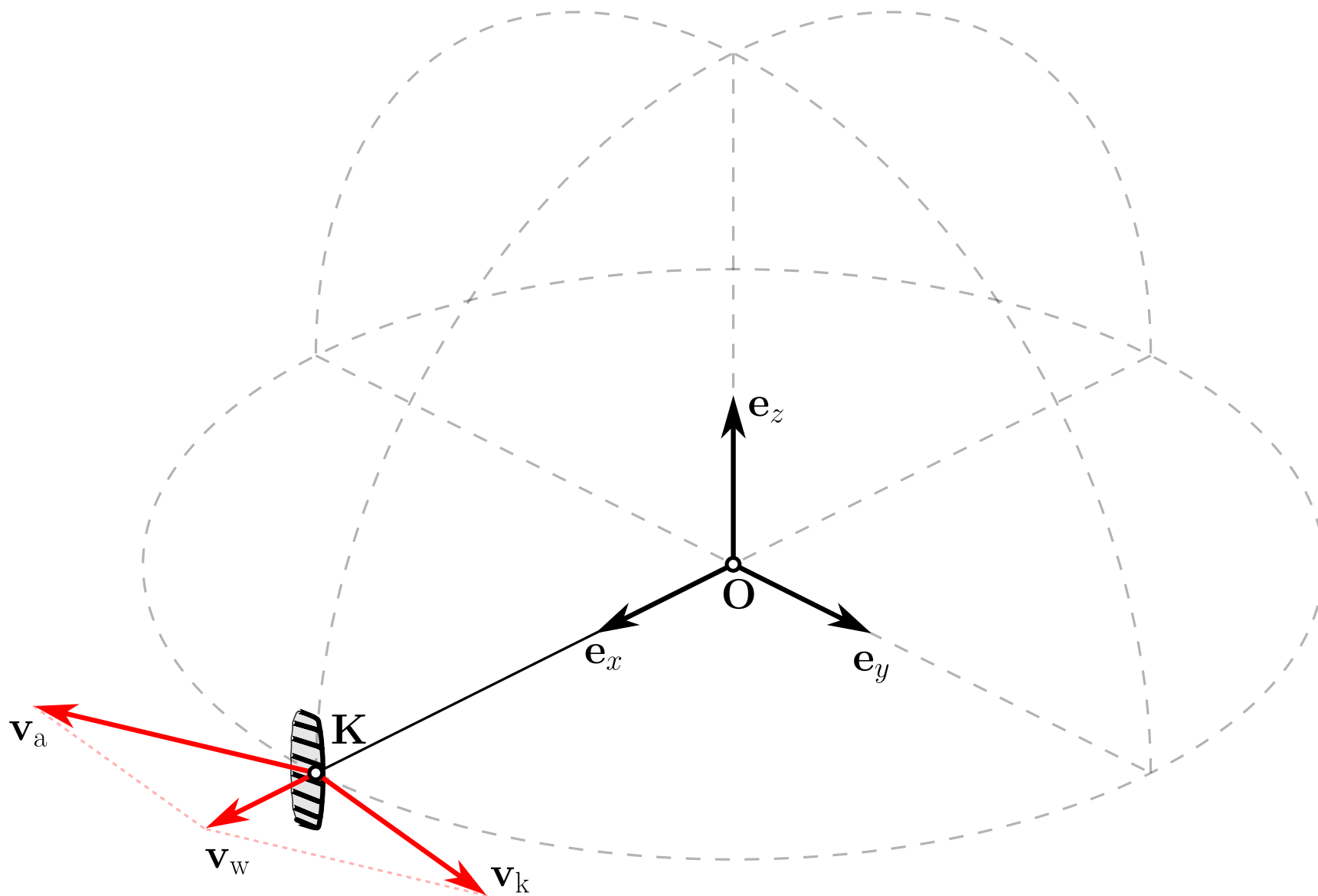


TU Delft V3 kite flying on launch mast in 2012





3D setting





Massless crosswind kite

Reeling factor $\leadsto$ degree of freedom

$$f = \frac{v_{k,x}}{v_w}$$

Tangential velocity factor

$$\lambda = \frac{v_{k,y}}{v_w}$$

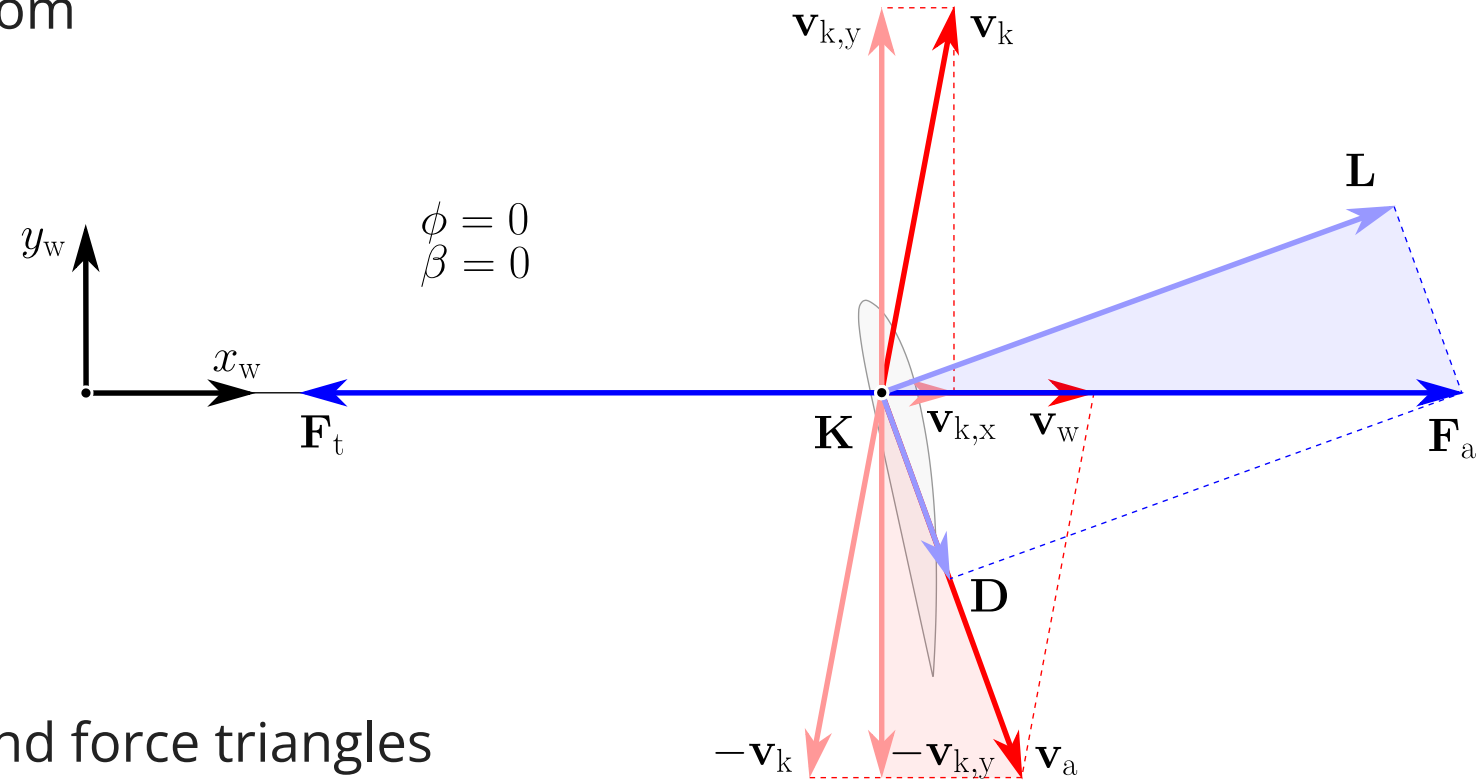
Apparent wind velocity

$$\begin{aligned} \mathbf{v}_a &= \mathbf{v}_w - \mathbf{v}_k \\ &= (v_w - v_{k,x}) \mathbf{e}_x + (-v_{k,y}) \mathbf{e}_y \end{aligned}$$

Geometric similarity of velocity and force triangles

$$\frac{L}{D} = \frac{v_{k,y}}{v_w - v_{k,x}} \quad \leadsto \quad \frac{L}{D} (v_w - v_{k,x}) = v_{k,y}$$

$$\lambda = \frac{L}{D} (1 - f) = E (1 - f) \quad \text{dependent variable, i.e. not a degree of freedom!}$$





Massless crosswind kite

Apparent wind velocity

$$v_a^2 = (v_w - v_{k,x})^2 + v_{k,y}^2 \quad \leadsto \quad \left(\frac{v_a}{v_w} \right)^2 = (1 - f)^2 + \lambda^2$$

$$\frac{v_a}{v_w} = (1 - f) \sqrt{1 + E^2}$$

Aerodynamic force

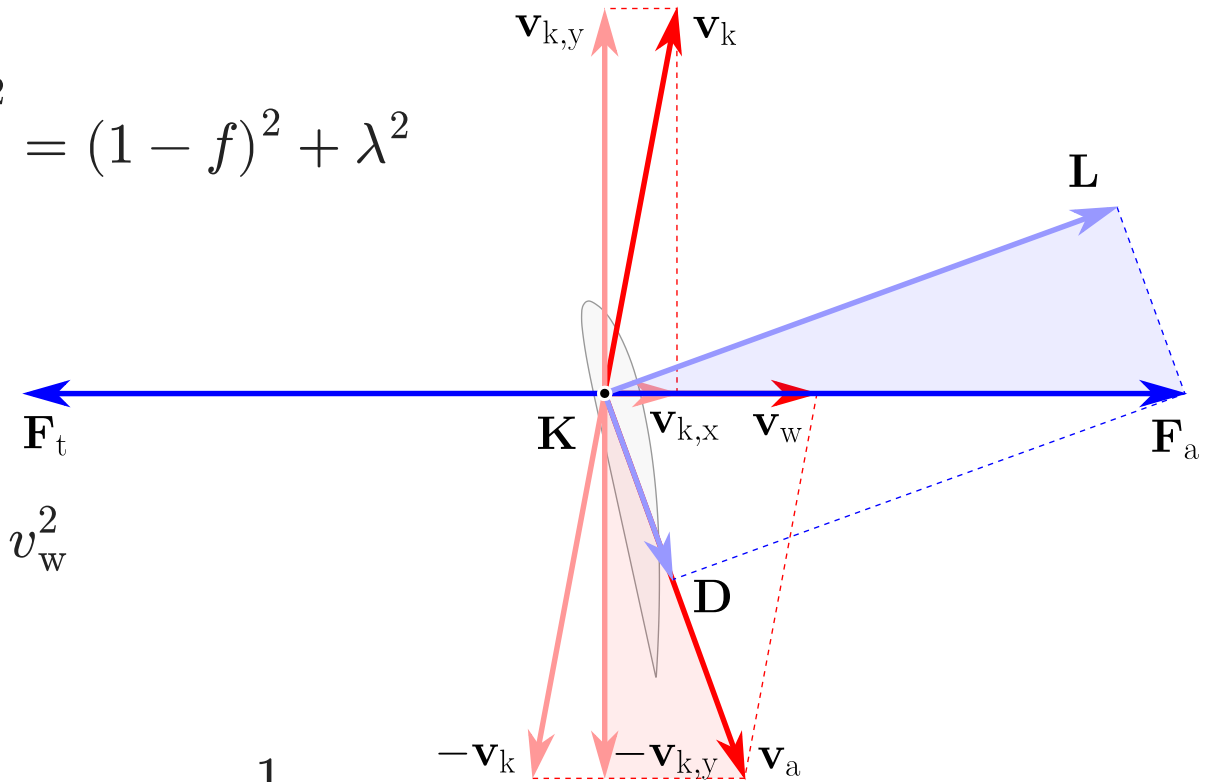
$$F_a = \frac{1}{2} \rho S C_R v_a^2 = \frac{1}{2} \rho S C_L \sqrt{1 + \frac{1}{E^2}} \left(\frac{v_a}{v_w} \right)^2 v_w^2$$

Non-dimensional tether force

$$\frac{F_t}{qS} = C_L \sqrt{1 + \frac{1}{E^2}} (1 - f)^2 (1 + E^2), \quad \text{with} \quad q = \frac{1}{2} \rho v_w^2$$

Power harvesting factor

$$\zeta = \frac{P}{P_w} = C_L \sqrt{1 + \frac{1}{E^2}} f (1 - f)^2 (1 + E^2), \quad \text{with} \quad P_w = \frac{1}{2} \rho v_w^3$$





Optimal reel-out speed

Maximum power harvesting factor at maximum value of $f(1 - f)^2$

This leads to a condition for the first derivative

$$0 = \frac{d}{df} [f(1 - f)^2]$$

from which the optimal reeling factor can be determined

$$f_{\text{opt}} = \frac{1}{3}$$

Optimal harvesting factor using f_{opt}

$$\zeta_{\text{opt}} = \frac{4}{27} C_L \sqrt{1 + \frac{1}{E^2}} (1 + E^2)$$



Reality check

The analytic formula for ζ_{opt} is based on a number of **idealizations**:

- ideal crosswind operation at $\beta = \phi = 0$,
- neglected reel-in phase,
- vanishing mass of the airborne system components,
- dragless tether.

In reality, the achievable power harvesting factor is **substantially lower** than predicted by ζ_{opt} .

Compare achievable power harvesting factor of the [Kitepower Falcon](#):

- rated power of 100 kW at the rated wind speed of 10 m/s,
- theoretical optimal power harvesting factor $\zeta_{\text{opt}} = 11.36$,
- achievable power harvesting factor $\zeta = 3.47$.
- This is about **30%** of the theoretical optimum value.

Check yourself



Reality check code

```
import numpy as np

kite = "Kitepower Falcon (V9)"
CL   = 1.19           # lift coefficient
CD   = 0.15           # drag coefficient
S    = 47             # projected surface area (m^2)
rho  = 1.225          # air density (kg/m^3)
vwr  = 10             # rated wind speed (m/s)
Pr   = 100            # rated power (kW)
E    = CL/CD          # lift-to-drag ratio
Pw   = 0.5*rho*vwr**3 # wind power density
Pr   = Pr*1000        # convert kW into W

zetaopt = 4/27*CL*np.sqrt(1+1/(E**2))*(1+E**2)
zetamax = 4/27*CL*E**2
zetareal = Pr/(Pw*S)

print('zetaopt = ', zetaopt)
print('zetareal = ', zetareal)
print('achieved = ', zetareal/zetaopt*100, '%')
print('zetamax = ', zetamax)
```

produces

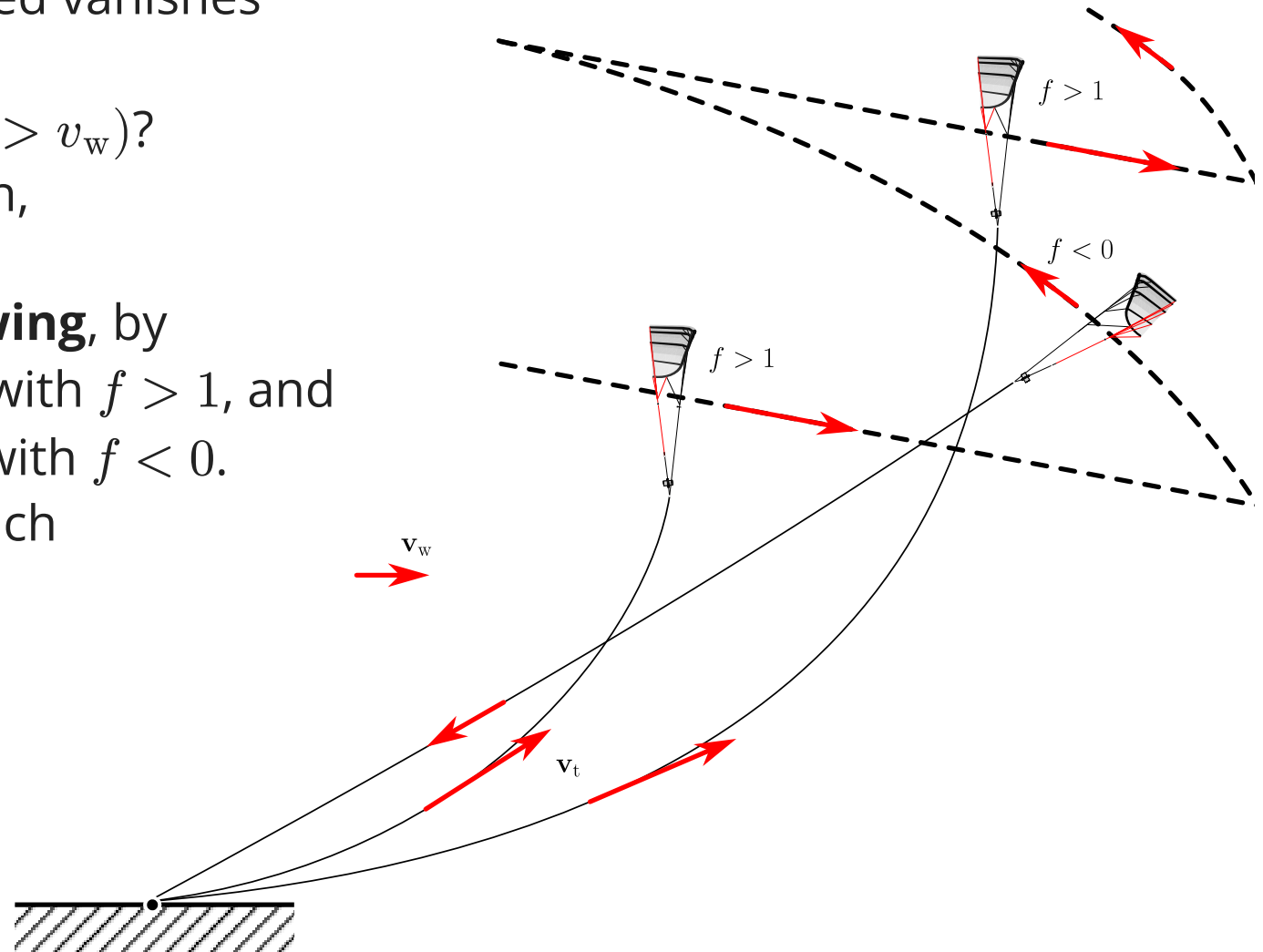
```
zetaopt = 11.361189218760718
zetareal = 3.4737299174989142
achieved = 30.575407649779727 %
zetamax = 11.095697119341564
```





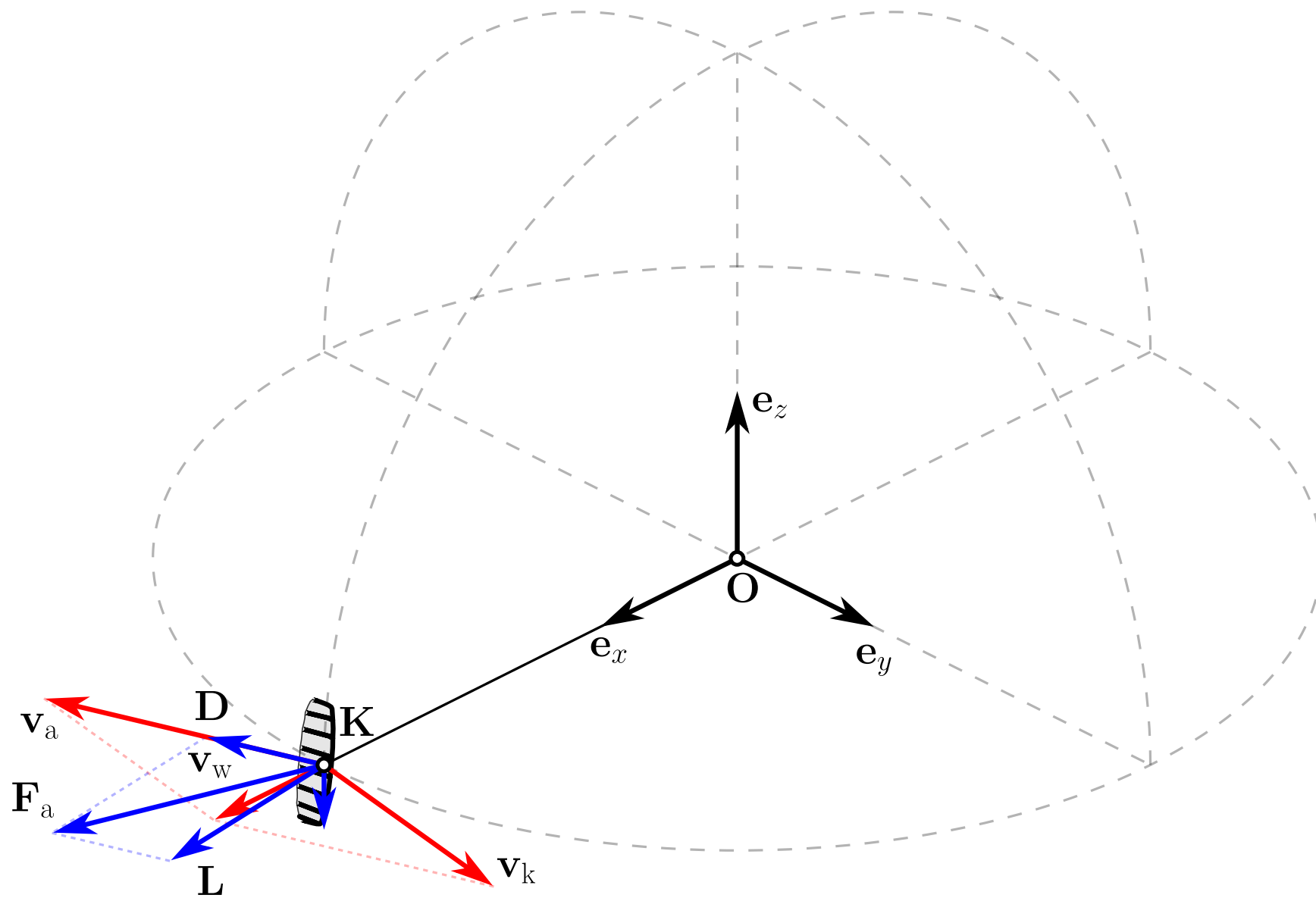
Maximum reel-out factor?

- At $f = 1$ the apparent wind speed vanishes
→ kite can not produce any lift.
- But what happens for $f > 1$ ($v_t > v_w$)?
→ gliding in downwind direction,
→ vanishing tether tension.
- Practical application is **step towing**, by alternating **downwind gliding** with $f > 1$, and **upwind retraction** of the kite with $f < 0$.
- Technique well known from winch launching of paragliders.



Note: for $\beta > 0$ and/or $\phi \neq 0$, v_a vanishes for $f < 1$.







Crosswind kite with mass

Static force equilibrium

$$\mathbf{F}_t + \mathbf{L} + \mathbf{D} + m\mathbf{g} = 0$$

Horizontal lift force component

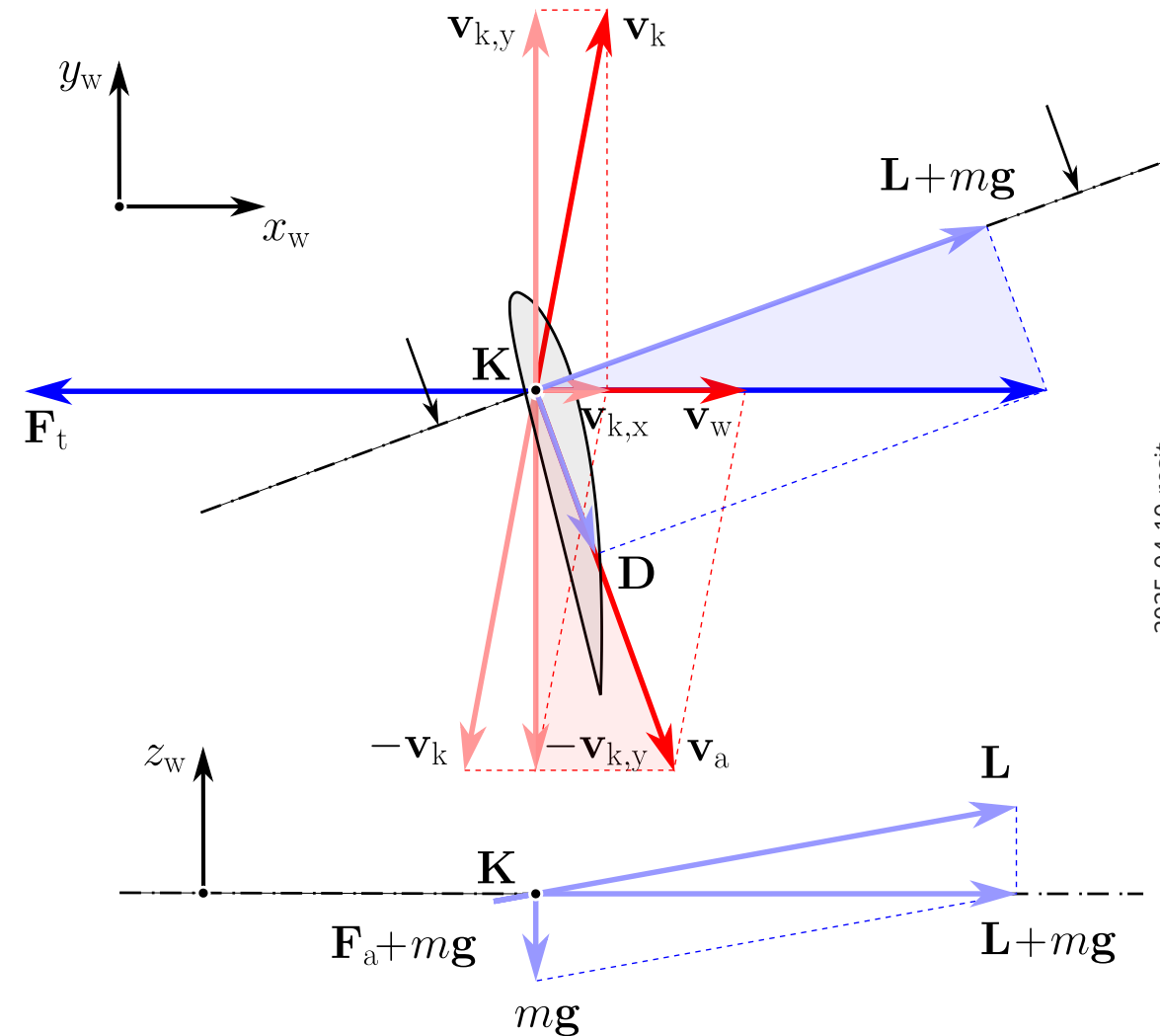
$$|\mathbf{L} + m\mathbf{g}| = \sqrt{L^2 - (mg)^2}$$

Geometric similarity

$$\frac{v_{k,y}}{v_w - v_{k,x}} = \frac{\sqrt{L^2 - (mg)^2}}{D}$$

Tangential velocity factor

$$\lambda = (1 - f)E\sqrt{1 - \left(\frac{mg}{L}\right)^2}$$



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Crosswind kite with mass

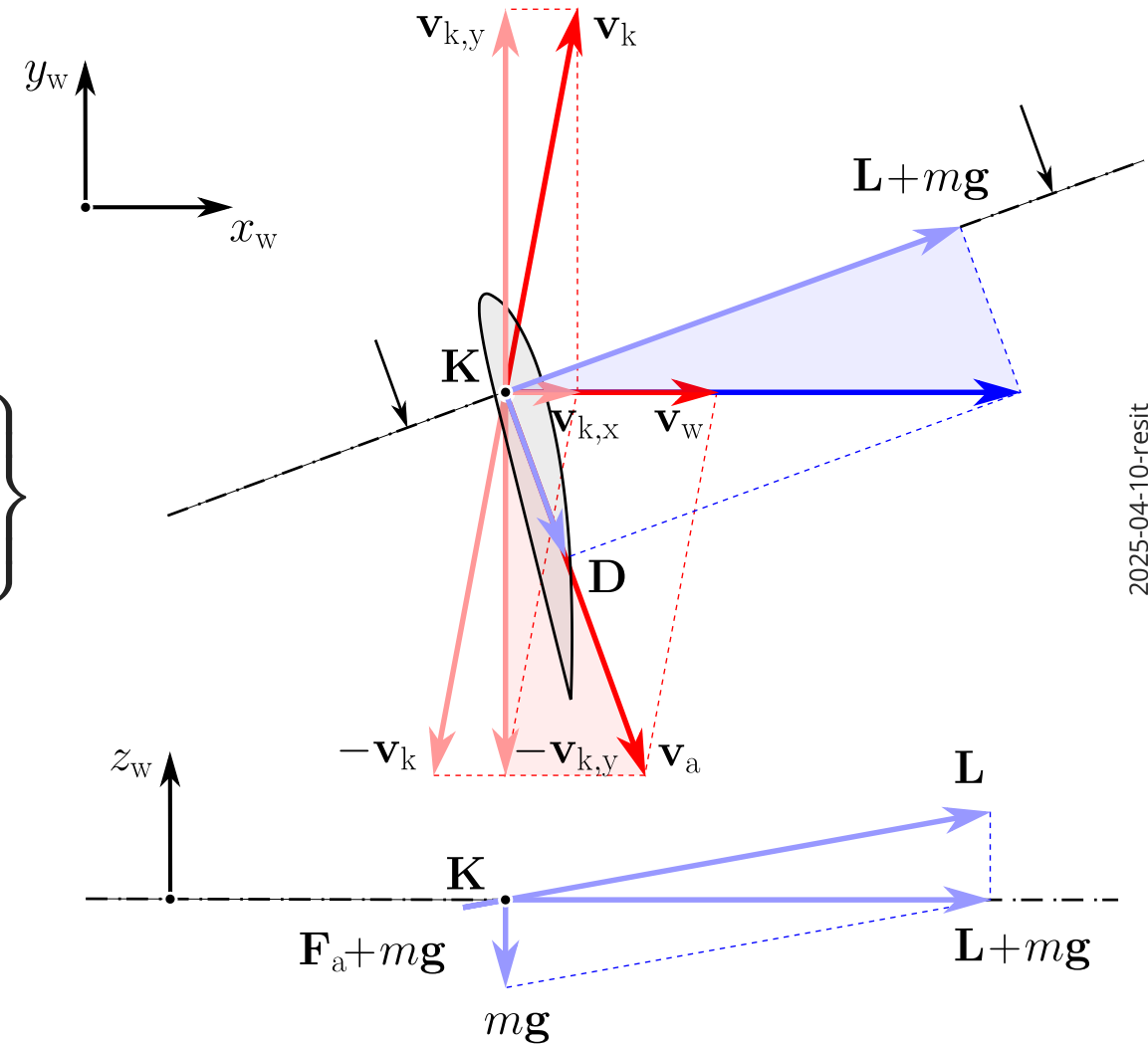
Apparent wind speed

$$\begin{aligned} \left(\frac{v_a}{v_w} \right)^2 &= (1 - f)^2 + \lambda^2 \\ &= (1 - f)^2 \left\{ 1 + E^2 \left[1 - \left(\frac{mg}{L} \right)^2 \right] \right\} \\ &= (1 - f)^2 \left\{ 1 + E^2 \left[1 - \left(\frac{\frac{2g}{\rho C_L} \frac{m}{S}}{v_a^2} \right)^2 \right] \right\} \end{aligned}$$

Substitute

$$x = \left(\frac{v_a}{v_w} \right)^2 \rightsquigarrow \rightsquigarrow$$

$$x = (1 - f)^2 \left\{ 1 + E^2 \left[1 - \left(\frac{v_{w,sto}}{v_w} \right)^4 \frac{1}{x^2} \right] \right\}$$



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Crosswind kite with mass

$$\textcolor{red}{x} = (1 - f)^2 \left\{ 1 + E^2 - E^2 \left(\frac{v_{\text{w,sto}}}{v_{\text{w}}} \right)^4 \frac{1}{\textcolor{red}{x}^2} \right\}$$

Cubic equation, with missing linear term

$$\textcolor{red}{x}^3 - (1 - f)^2 (1 + E^2) \textcolor{red}{x}^2 + (1 - f)^2 E^2 \left(\frac{v_{\text{w,sto}}}{v_{\text{w}}} \right)^4 = 0$$

Cubic equation in standard form

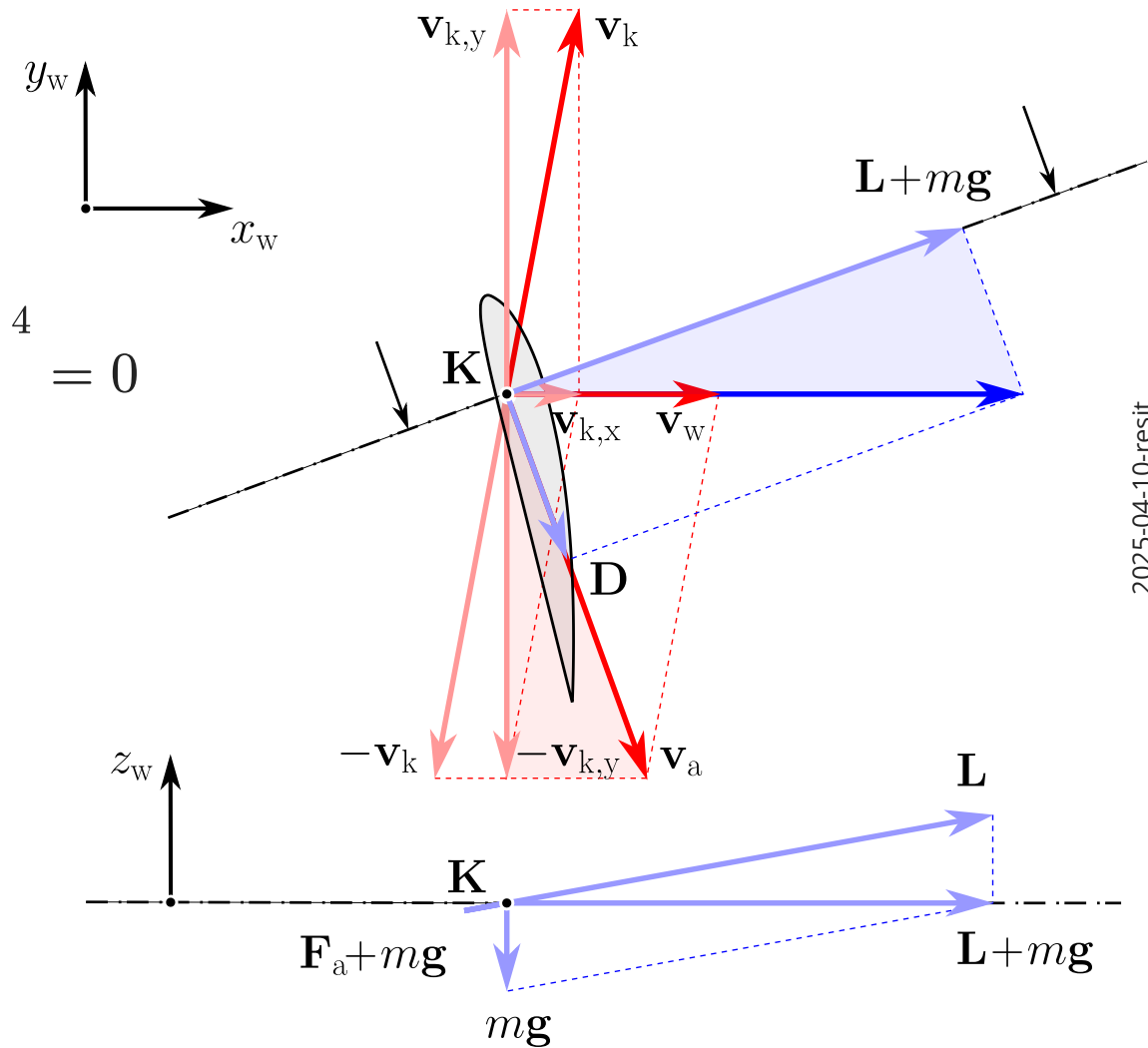
$$ax^3 + bx^2 + cx + d = 0, \quad \text{with}$$

$$a = 1,$$

$$b = -(1 - f)^2 (1 + E^2) ,$$

$$c = 0,$$

$$d = (1 - f)^2 E^2 \left(\frac{v_{\text{w,sto}}}{v_{\text{w}}} \right)^4.$$





Number and nature of roots

A cubic equation has three roots. The nature of the three roots depends on the value of the discriminant

$$\begin{aligned}\Delta &= \cancel{18abcd} - 4b^3d + \cancel{b^2c^2} - \cancel{4ac^3} - 27a^2d^2, \quad \text{because } c = 0 \\ &= 4(1-f)^8(1+E^2)^3E^2 \left(\frac{v_{w,\text{sto}}}{v_w} \right)^4 - 27(1-f)^4E^4 \left(\frac{v_{w,\text{sto}}}{v_w} \right)^8, \\ &= (1-f)^4E^2 \left(\frac{v_{w,\text{sto}}}{v_w} \right)^4 \left[4(1-f)^4(1+E^2)^3 - 27E^2 \left(\frac{v_{w,\text{sto}}}{v_w} \right)^4 \right]\end{aligned}$$

For $\Delta < 0$, the equation has one real root and two non-real complex conjugate roots.

For $\Delta = 0$, the equation has one real root and two real repeated roots.

For $\Delta > 0$, the equation has three distinct real roots.



Cubic polynomial

Polynomials for the TU Delft V3 kite at $f = 0$.

Polynomial function and derivatives

$$p(x) = x^3 - (1 - f)^2 (1 + E^2) x^2 + (1 - f)^2 E^2 \left(\frac{v_{w,sto}}{v_w} \right)^4,$$

$$p'(x) = 3x^2 - 2(1 - f)^2 (1 + E^2) x,$$

$$p''(x) = 6x - 2(1 - f)^2 (1 + E^2).$$

→ Wind speed v_w affects vertical offset of polynomial

End behavior

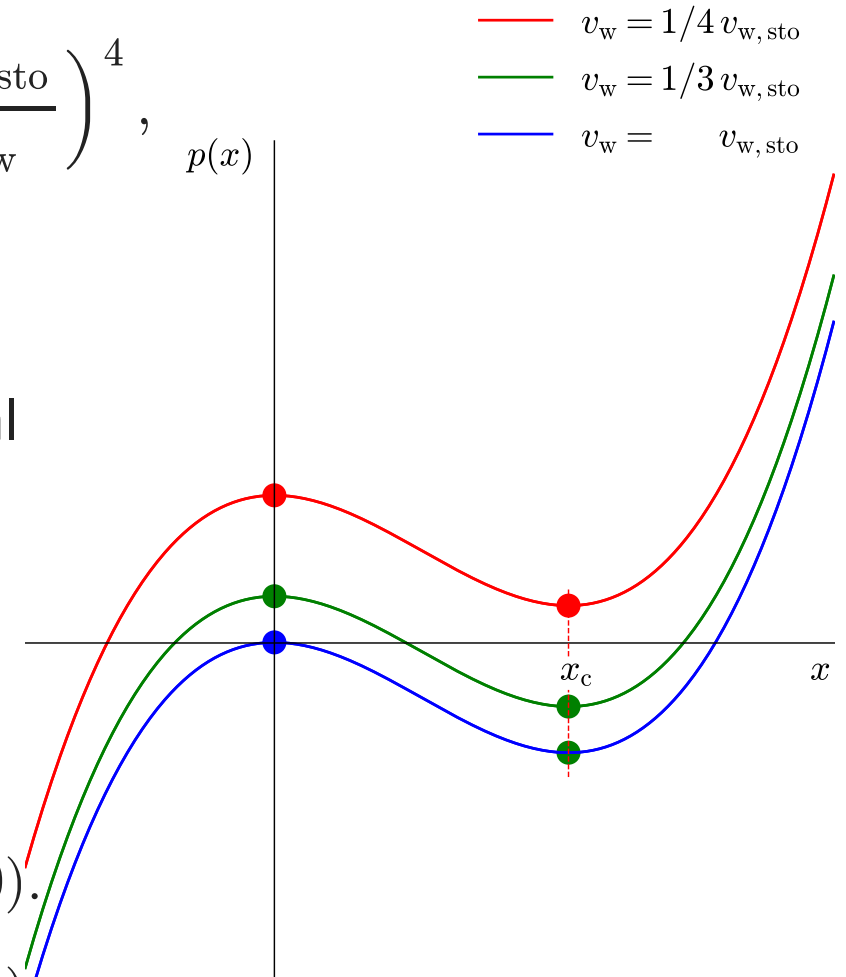
$$x \rightarrow -\infty \leadsto p(x) \rightarrow -\infty,$$

$$x \rightarrow \infty \leadsto p(x) \rightarrow \infty.$$

Critical points: local extrema ($p' = 0$)

$$x = 0 \leadsto \text{local maximum } (p'' < 0).$$

$$x_c = \frac{2}{3}(1 - f)^2 (1 + E^2) \leadsto \text{local minimum } (p'' > 0).$$





Physical solution space

$$x \geq x_{\min} = (1 - f)^2.$$

This lower limit follows from the [definition](#) of the non-dimensional apparent wind speed as

$$x = (1 - f)^2 + \lambda^2,$$

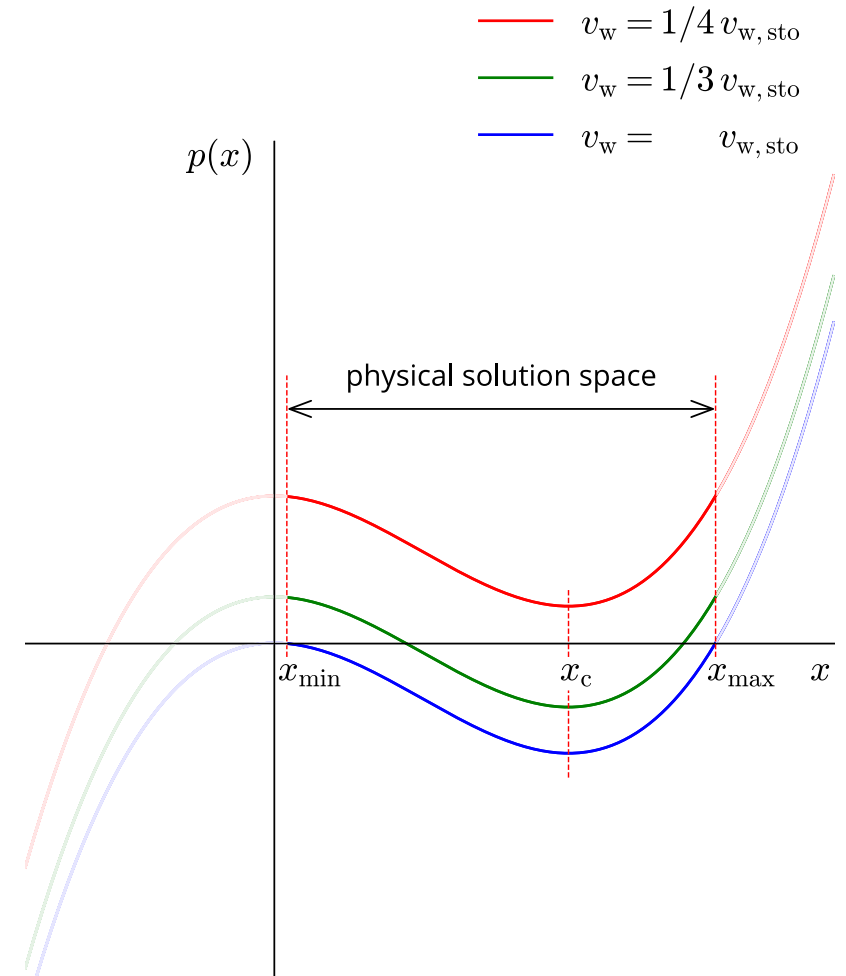
and the fact that the crosswind contribution is always positive, i.e. $\lambda^2 \geq 0$.

$$x \leq x_{\max} = (1 - f)^2 (1 + E^2).$$

This upper limit is the apparent wind speed for Loyd's [ideal](#) crosswind case, where $m = 0$.

Interesting is also the observation $x_c = \frac{2}{3} x_{\max}$.

Polynomials for the TU Delft V3 kite at $f = 0$.

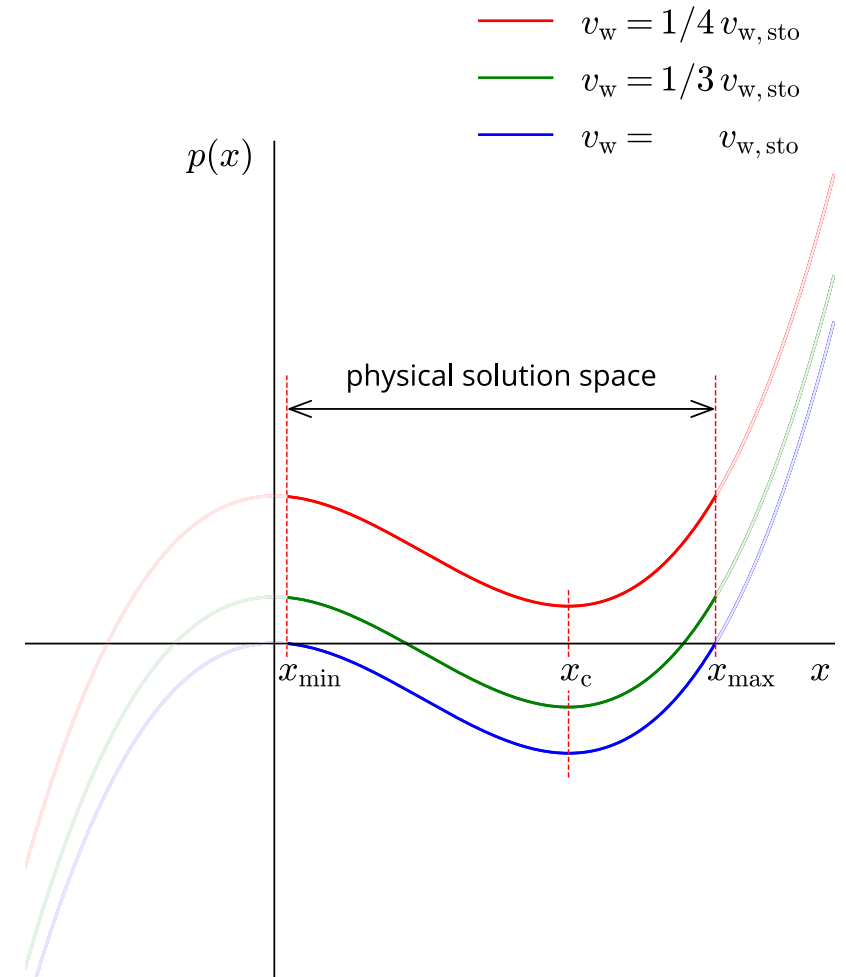




Minimum flyable wind speed

- At $v_w = \frac{1}{4}v_{w,sto}$, the cubic equation has **no real roots** in the physical solution space.
- At $v_w = \frac{1}{3}v_{w,sto}$, the cubic equation has **two real roots** in the physical solution space.
- At what value of v_w does the cubic equation have a **repeated real root**, transitioning from no real root to two real roots?
- This transition is described mathematically by the discriminant Δ changing the sign from negative to positive.

Polynomials for the TU Delft V3 kite at $f = 0$.





Discriminant

The discriminant is positive when

$$4(1 - f)^4(1 + E^2)^3 > 27E^2 \left(\frac{v_{w,sto}}{v_w} \right)^4,$$

$$\left(\frac{v_w}{v_{w,sto}} \right)^4 > \frac{27E^2}{4(1 - f)^4(1 + E^2)^3},$$

$$v_w > \sqrt[4]{\frac{27E^2}{4(1 + E^2)^3} \frac{v_{w,sto}}{1 - f}},$$

$$v_w > \frac{v_{w,dto}}{1 - f}, \quad \text{with}$$

$$v_{w,dto} = \sqrt[4]{\frac{27E^2}{4(1 + E^2)^3}} v_{w,sto}, \quad \text{and}$$

$$v_{w,sto} = \sqrt{\frac{2g}{\rho C_L} \frac{m}{S}}.$$

► But what is the physical meaning of $v_{w,dto}$?





Cardano's method

```
def solve_cubic_equation(a, b, c, d):  
    '''Solves the general cubic equation defined by coefficients a, b, c and d  
    ...  
    # Coefficients of the depressed cubic equation  
    p = (3*a*c - b**2) / (3*a**2)  
    q = (2*b**3 - 9*a*b*c + 27*a**2*d) / (27*a**3)  
    # Discriminant  
    delta = - q**2 / 4 - p**3 / 27  
    # Number and type of roots  
    if delta < 0: # One real root (x1) and two complex roots (x2, x3)  
        u = math.cbrt(-q/2 + math.sqrt(-delta))  
        v = math.cbrt(-q/2 - math.sqrt(-delta))  
        x1 = -(u + v)/2 - b / (3*a) + (u - v)*cmath.sqrt(3)/2j  
        x2 = u + v - b / (3*a) # This is the real root!  
        x3 = -(u + v)/2 - b / (3*a) - (u - v)*cmath.sqrt(3)/2j  
    elif delta == 0: # Three real roots, two are equal  
        u = (-q/2)**(1/3)  
        v = u  
        x1 = 2*u - b / (3*a)  
        x2 = x1  
        x3 = -u - b / (3*a)  
    else: # Three distinct real roots  
        u = (-q/2 + cmath.sqrt(-delta))**(1/3)  
        v = (-q/2 - cmath.sqrt(-delta))**(1/3)  
        x1 = u + v - b / (3*a)  
        x2 = -(u + v)/2 - b / (3*a) + (u - v)*cmath.sqrt(3)/2j  
        x3 = -(u + v)/2 - b / (3*a) - (u - v)*cmath.sqrt(3)/2j  
    return x1, x2, x3
```



Cubic polynomial

Polynomial for $f = 0$ and physical parameters of the TU Delft V3 kite

Investigate the cubic polynomial for three different wind speeds,

$v_w = 0.9v_{w,dto}$, $v_{w,dto}$ and $1.1v_{w,dto}$.

By definition $x = \left(\frac{v_a}{v_w}\right)^2 \geq 0$,

$\leadsto x < 0$ unphysical.

For $v_w < v_{w,dto}$ ($\Delta < 0$),

$\leadsto$ none of the real roots in physical range.

For $v_w = v_{w,dto}$ ($\Delta = 0$),

$\leadsto$ repeated real root.

For $v_w > v_{w,dto}$ ($\Delta > 0$).

$\leadsto$ two distinct real roots.

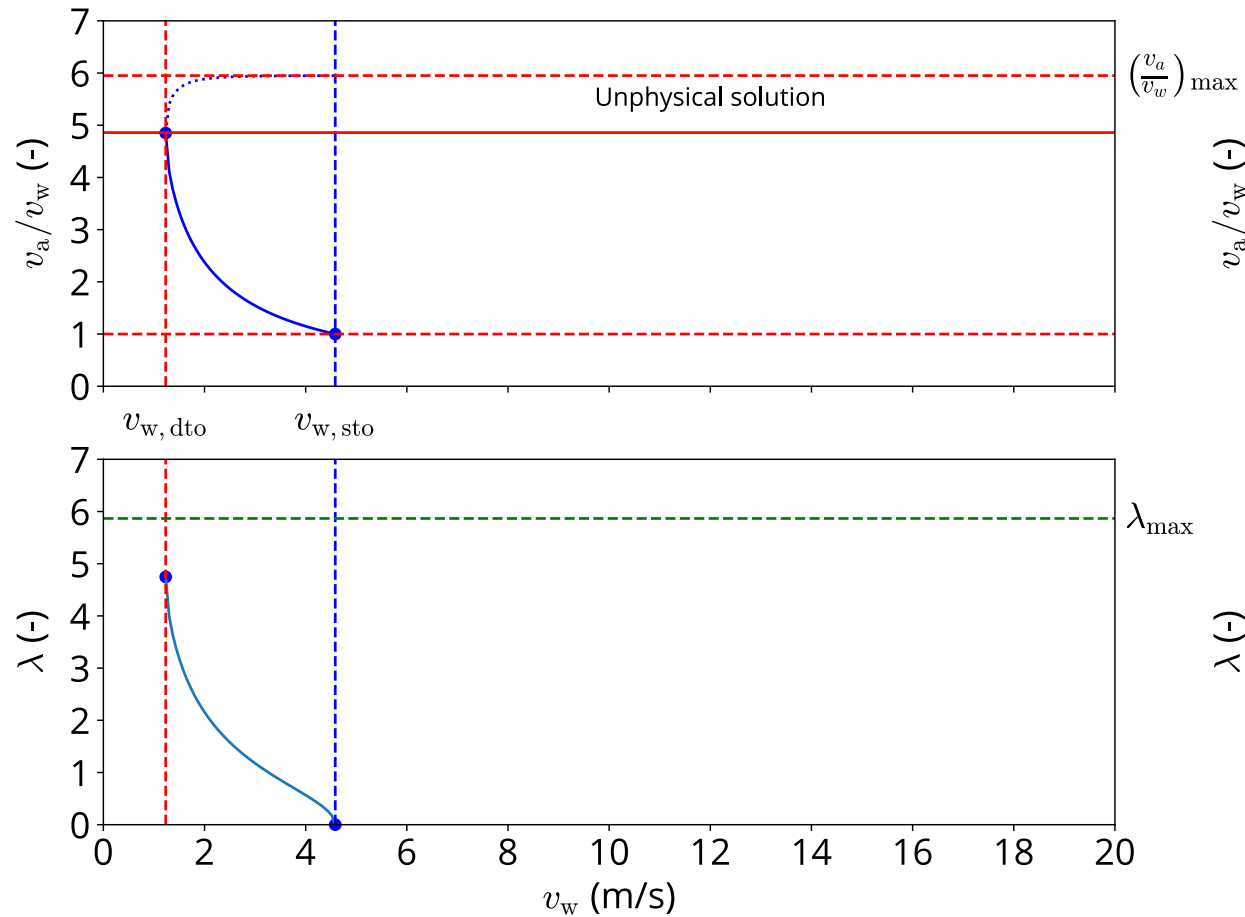
► Only interested in the lower value.





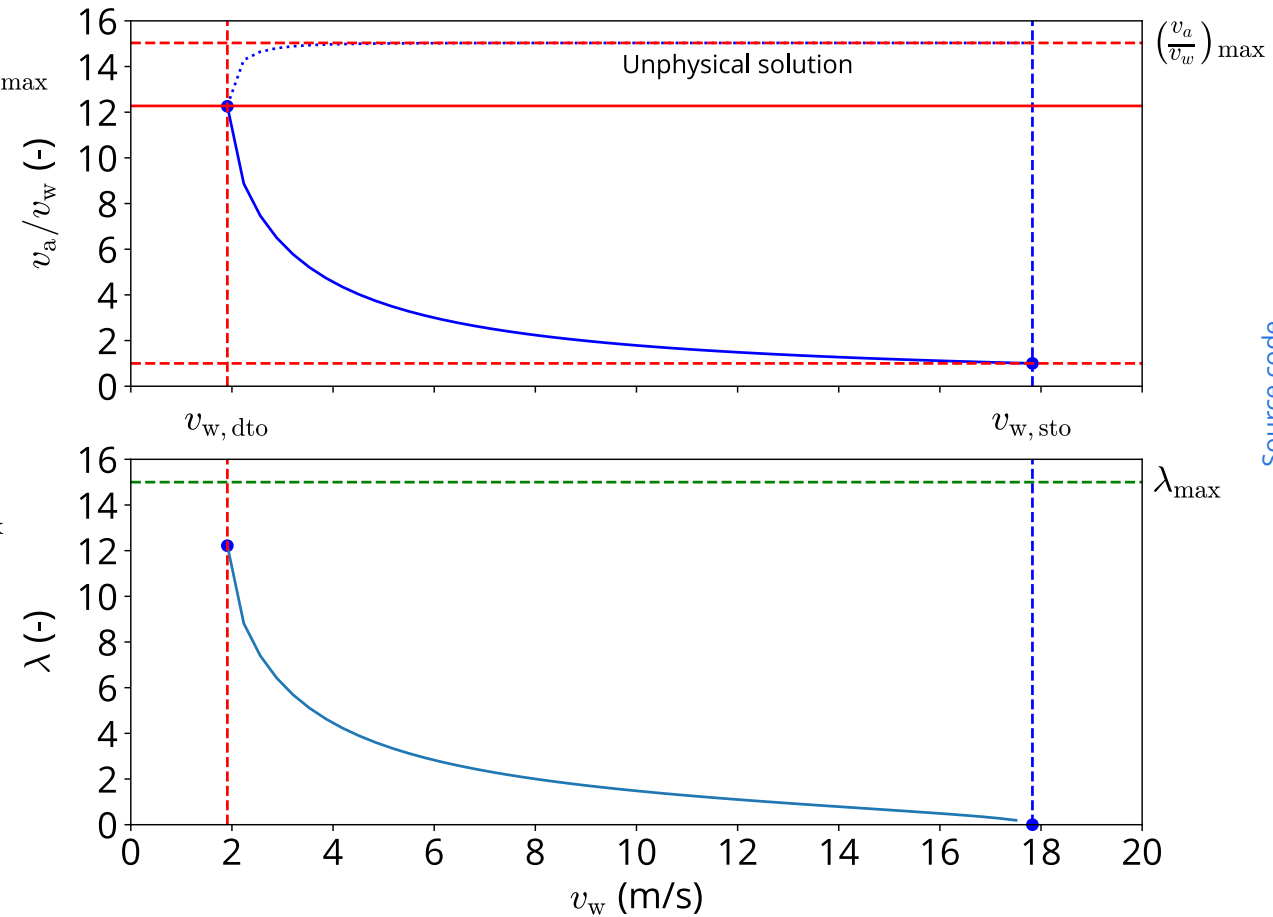
Full wind speed range

TU Delft V3



Tangential velocity factor $\lambda_{max} = \lambda(m = 0) = E$.

Mozaero AP3



Source code





Dynamic take-off wind speed





Interpretation

Considering a kite with a **static take-off wind speed**

$$v_{w,sto} = \sqrt{\frac{2g}{\rho C_L} \frac{m}{S}},$$

the **dynamic take-off wind speed** for crosswind operation at $\beta = \phi = 0$ is given by

$$v_{w,dto} = \sqrt[4]{\frac{27E^2}{4(1+E^2)^3}} v_{w,sto}.$$

- This definition of a lower wind speed limit implies a constant tether length ($f = 0$).
- For $v_w < v_{w,dto}$, the kite can not generate a sufficient lift force for sustained flight.
- For $v_{w,dto} < v_w < v_{w,sto}$, the generated lift force can counteract gravity. In this wind speed regime, the tangential velocity factor λ is a function of the wind speed v_w .
- For $v_w > v_{w,sto}$, the generated lift force is too high, driving the kite to a force equilibrium at higher elevation angles β , or requiring reeling out ($f > 0$).



Interpretation

- By reeling in ($f < 0$), the flyable wind speed range can be temporarily decreased below the value of $v_{w,dto}$, however, this condition can not be maintained for longer time.
- This low wind speed regime is particularly important for dynamic launching of kites.
- While the static take-off wind speed $v_{w,sto}$ is affected by the lift coefficient C_L , the dynamic take-off wind speed $v_{w,dto}$ adds the dependence on the lift to drag ratio E .
- Static take-off wind speed is affected by mass scaling, predominantly via the ratio m/S .
- Dynamic take-off wind speed is substantially lower, especially for kites with high aerodynamic performance $E = L/D$.
- The analysis is highly idealized and the real cut-in wind speed of the kite will be higher.



Approximation

For $E \gg 1$ the expression for $v_{w,dto}$ can be approximated

$$\begin{aligned} v_{w,dto} &= \sqrt[4]{\frac{27E^2}{4(1+E^2)^3}} v_{w,sto}, \\ &= \sqrt[4]{\frac{27}{4}} \frac{1}{E} v_{w,sto}, \\ &= \frac{1.612}{E} v_{w,sto}. \end{aligned}$$





Massless radial kite

Reeling factor $\leadsto$ degree of freedom

$$f = \frac{v_k}{v_w}$$

Apparent wind velocity

$$\mathbf{v}_a = \mathbf{v}_w - \mathbf{v}_k$$

Decomposition of kite velocity

$$\mathbf{v}_k = \mathbf{b} + \mathbf{c}$$

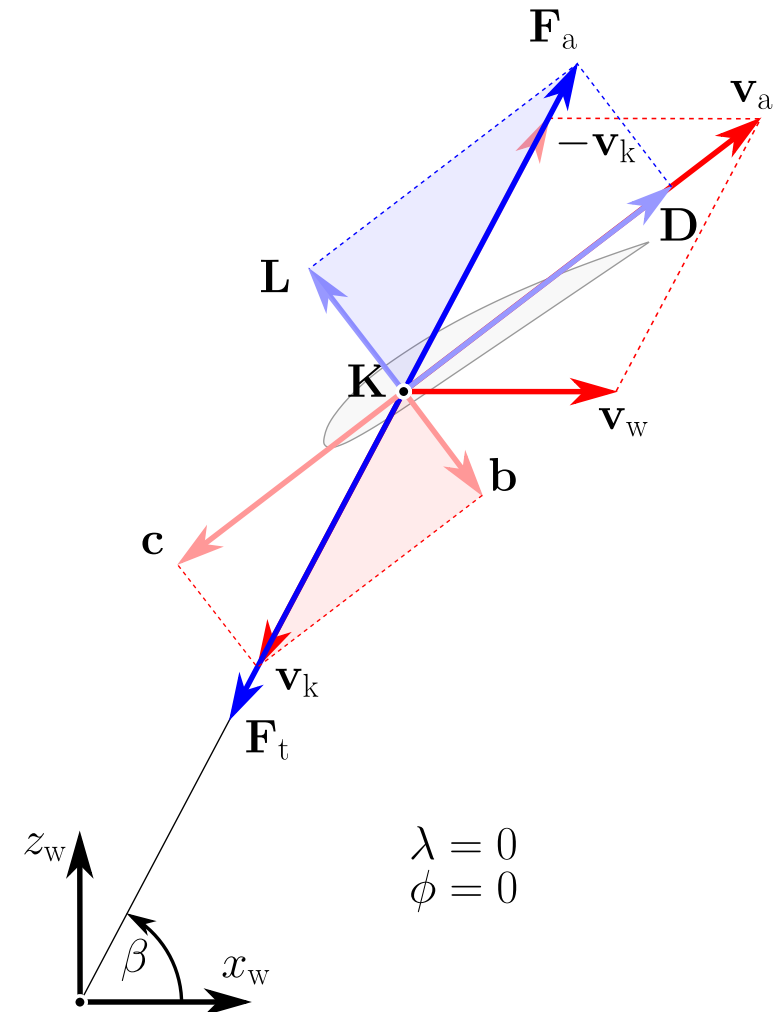
$$\mathbf{v}_a = \mathbf{v}_w - \mathbf{b} - \mathbf{c}$$

$$\mathbf{v}_w = \mathbf{v}_a + \mathbf{c} + \mathbf{b}$$

$$v_w^2 = (v_a - c)^2 + b^2 \quad (\text{positive vector magnitudes!})$$

$$\frac{v_a}{v_w} = \sqrt{1 - \left(\frac{b}{v_w}\right)^2} + \frac{c}{v_w}$$

Displayed is the case for reel in with $f < 0$




$$\frac{b}{v_k} = \frac{L}{F_a} = \frac{1}{\sqrt{1 + \frac{1}{E^2}}}$$

$$\frac{c}{v_k} = \frac{D}{F_a} = \frac{1}{\sqrt{1 + E^2}}$$

$$\frac{b}{v_{\mathrm{w}}} = - \frac{f}{\sqrt{1 + \frac{1}{E^2}}}$$

$$\frac{c}{v_w} = - \frac{f}{\sqrt{1 + E^2}}$$

Minus signs accounts for negative value of f

[illegible]

$$\begin{array}{l} \lambda = 0 \\ \phi = 0 \end{array}$$



Massless radial kite

Apparent wind speed

$$\frac{v_a}{v_w} = \sqrt{1 - \frac{f^2}{1 + \frac{1}{E^2}}} - \frac{f}{\sqrt{1 + E^2}},$$

$$= \frac{\sqrt{1 + E^2(1 - f^2)} - f}{\sqrt{1 + E^2}}$$

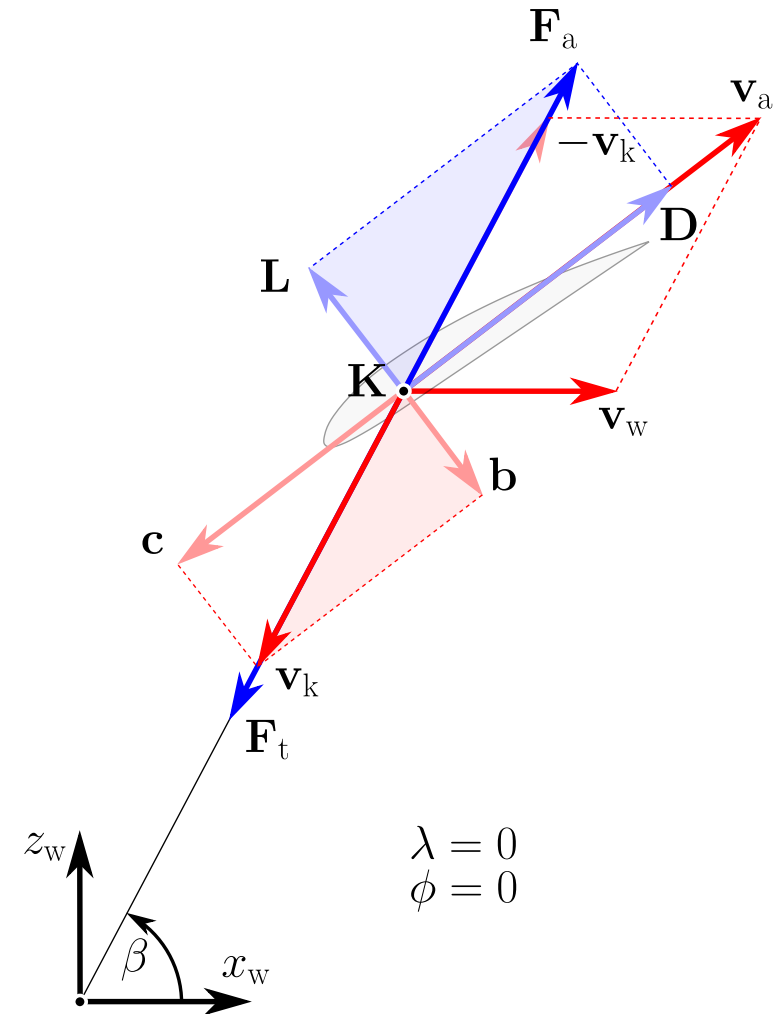
Non-dimensional tether force

$$\frac{F_t}{qS} = C_L \frac{[\sqrt{1 + E^2(1 - f^2)} - f]^2}{E\sqrt{1 + E^2}}$$

Power harvesting factor

$$\zeta = C_L \frac{f [\sqrt{1 + E^2(1 - f^2)} - f]^2}{E\sqrt{1 + E^2}}$$

Displayed is the case for reel in with $f < 0$





Massless radial kite

Elevation angle $\leadsto$ not a degree of freedom

$$\cos \beta = \frac{\sqrt{1 + E^2(1 - f^2)} + fE^2}{1 + E^2}$$

Homework #1: Derive this expression.

Hint:

- Formulate two different expressions for the apparent wind speed, one of them containing $\cos \beta$.
- Eliminate the apparent wind speed by equating those equations.
- Solve for $\cos \beta$.



Massless radial kite

Minimum reeling factor

$$f_{\min} = -\sqrt{1 + \frac{1}{E^2}}$$

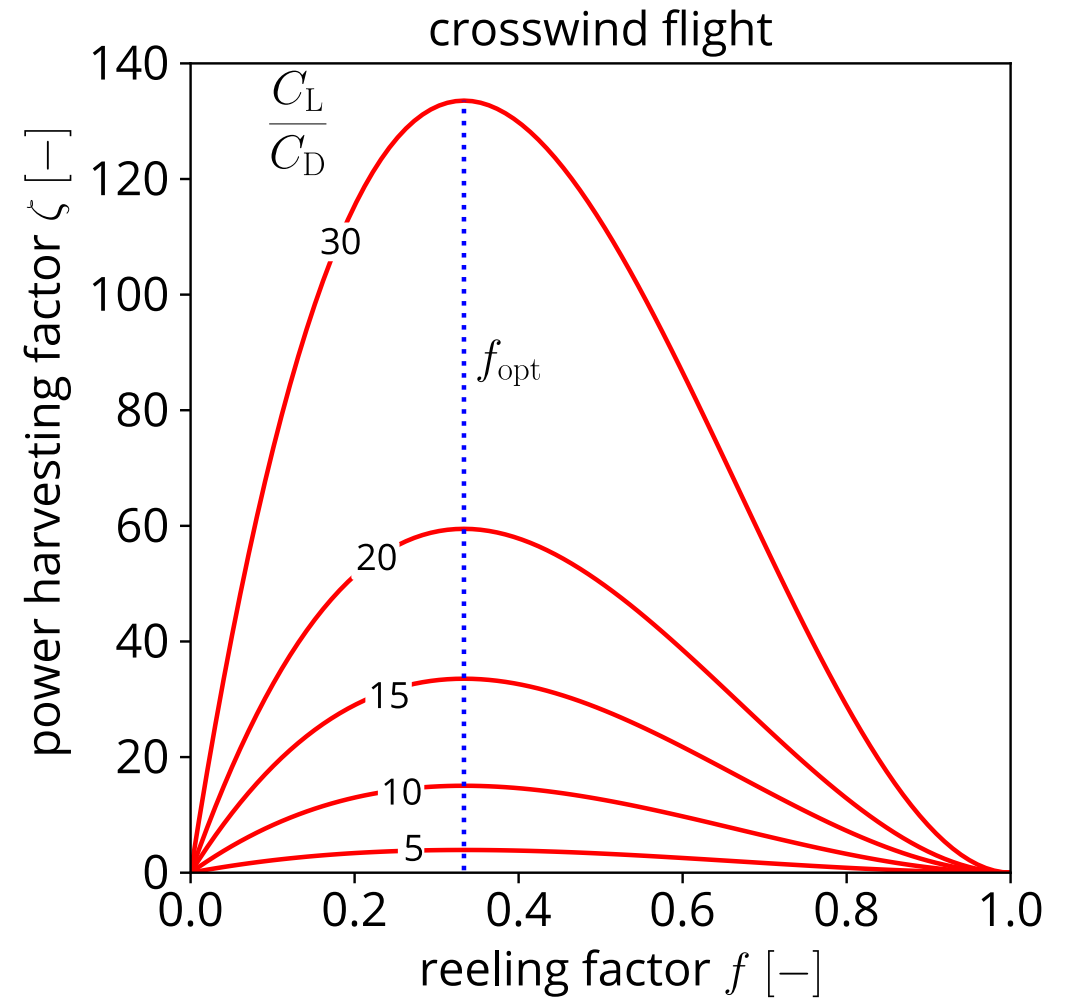
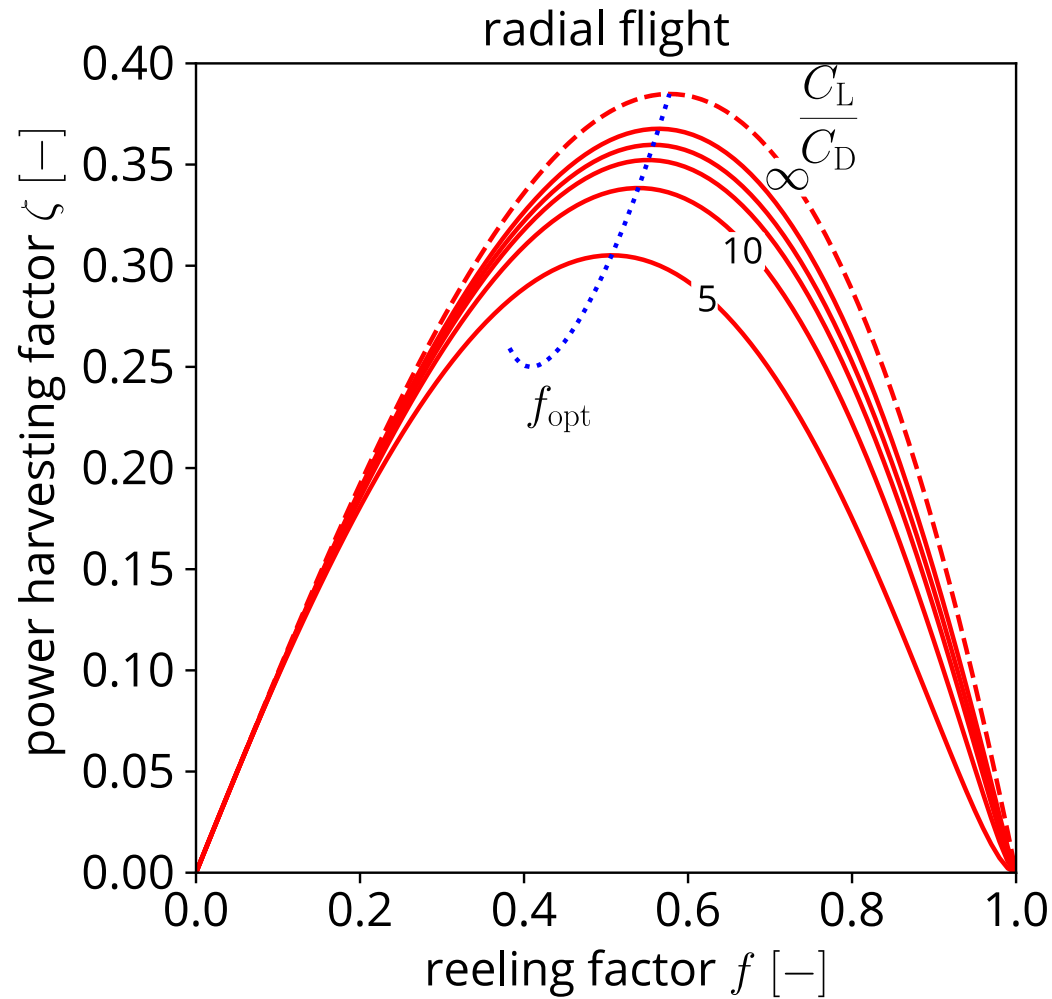
Homework #2:
Derive this expression.

Hint:

- Use the fact that the radicant in the expression for $\cos \beta$ has to be greater or equal to zero.



Loyd's theory - comparison





How to use these analytic models?

- ▶ The wind speed v_w describes the available energy resource.
- ▶ The lift-to-drag ratio E characterizes the specific kite design.

For the ideal crosswind kite prescribe

- ▶ the tether reeling speed v_t to calculate the achievable kite crosswind speed $v_{k,x}$, or
- ▶ the kite crosswind speed $v_{k,x}$ to calculate the required tether reeling speed v_t .

For the non-maneuvering kite prescribe

- ▶ the tether reeling speed v_t to calculate the achievable elevation angle β , or
- ▶ the elevation angle β to calculate the required tether reeling speed v_t .



References

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Questions?

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