

ALGORITHMIC GAME THEORY

GENERAL CONCEPTS

PREFERENCE LIST: \succsim_i COMPLETE + TRANSITIVE
 UTILITY FUNCTION: $g_i: \Theta \rightarrow \mathbb{R}: g_i(a) > g_i(b) \Leftrightarrow a \succsim_i b$
 RATIONALITY: MAXIMIZING $g = u$
 DOMINANT STRATEGY: BETTER PAYOFFS NO MATTER \succsim_{-i} STRATEGIES
 BEST RESPONSE: GIVEN S_{-i} , OPTIMIZE S_i N.E. IF MUTUAL $\begin{cases} \hat{p} = A \max p^T A c \\ \hat{q} = A \max p^T B c \end{cases}$

SOCIAL CHOICE THEORY

CHOICE FN: $g: \Pi^m \rightarrow [K]$ CANDIDATE
 WELFARE FN: $g: \Pi^m \rightarrow \Pi$ LIST ROBUST?: NEWBIE OR WIN!
 RATIONAL: $\bar{\lambda}$ IFF $0^k > 0 \forall 0 \in S \parallel$ IIA OR COST INTENCY IF $g(B) \in A \Rightarrow g(A) = g(B)$
 STRATEGICALLY MANIPUL: $\exists \{ \succsim_1, \dots, \succsim_m \}; \succsim_i: g(\gamma) = a \wedge g(\gamma') = b$
 CHOICE NOT MANIPULABLE: INCENTIVE COMPATIBLE
 UNANIMOUS WELFARE: $\forall i: a \succsim_i b \Rightarrow g(a, b) = a$
 CONSISTENT WELFARE: RELAT. POS IN g ARE INFLUENCED ONLY BY REL IN \succsim_i
 $\exists \{ \succsim_1, \dots, \succsim_m \}, \{ \succsim'_1, \dots, \succsim'_m \}: a \succsim_i b \Leftrightarrow a \succsim'_i b \forall i$
 $\Rightarrow g(\gamma) = g(\gamma') = a$ (AGREE)

VOTING SYSTEM: PLURALITY (1ST) BORDA (SCORE) MAJORITY (RANKING)
 CONDORCET: WIN ANY PAIRWISE COMPARISON ? \exists : NOT NEC...
 REVERSAL PARADOX: SAME WINNER, REV. PREFS

ARROW THM: ANY WELFARE UNANIMOUS AND CONSISTENT IS A DICTATORSHIP
 GIBB. SAT. : ANY CHOICE WHICH IS INCENTIVE COMPATIBLE IS A DICTATORSHIP.

THE VCG MECHANISM

OBJECTIVE $\max_{\bar{s}} \sum_{i \in O} v_i(s) = \max_{\bar{s}} \sum_{i \in O} b_i(s)$ MECHANISM IS A PLAYER!
 BIDS INSTEAD OF TRUE VALUATIONS $S_W = \sum_{i \in W} v_i$
 DSIC: ALWAYS TELL THE TRUTH! INDIV. RATIONALITY $b_i = v_i \forall i$
 CLARKE PINOTRULE: PRICE IS THE DAMAGE $\bar{\lambda}$ DOES BY PARTICIPATING
 $P_i = \text{WELFARE(AUCTION}(\bar{\lambda}_i)) - (\text{WELFARE(AUCTION)} - w_i)$
 NO POSITIVE TRANSFERS WITH CLARKE.

NASH EQUILIBRIA

NASH: ANY GAME WITH FINITE STRATEGIES + PLAYERS HAS A MNE!
 COROLL: AN MNE IS A BEST-R IFF PURE IN ITS SUPPORT ARE B-!
 MNE: $\pi_i(s_i) \cdot p(s_i) \geq \pi_i(\hat{s}_i, s_{-i}) \cdot p(\hat{s}_i, s_{-i})$
 C-NE: CORRELATION DEVICE $\pi_i(s_i) \cdot p(s_i) \geq \pi_i(\hat{s}_i, s_{-i}) \cdot p(s_i) \parallel$ IS
 COARSE CORRELATED NO ASSUMPTION OF KNOWING p JUST OBEY!

PAYOFF SPACE: CE POLYTOPE CONTAINS NE-POLY (SHARE VERTICES)

MINMAX: FOR ZERO-SUM GAMES
 $\begin{cases} \text{SAFE} & A \max_x [\min_y x^T A y] \\ \text{THREAT} & A \min_x [\max_y x^T A y] \end{cases}$
 // AND // OVERLAP AT NE
 LINEAR PROGRAM
 $\begin{cases} \text{SAFE} & A \max_x [\min_y x^T A y] \\ \text{THREAT} & A \min_x [\max_y x^T A y] \end{cases}$
 $\max \theta = \min \phi$
 S.T. $x^T A \geq \theta$ S.T. $A y \leq \phi$
 $\sum x_i = 1$ $\sum y_i = 1$ NO ADV: FOR FIRST MOVER!

LEMKE: A, B PAYOFF MATRICES
 HOWSON $Ax \leq 1 \mid Bx \leq 1$ LOOP
 $x \geq 0 \mid x \geq 0$
 SYMMETRIC $x^T (A - B) x = 0$
 4. RELAX
 B. MAKE ACTIVE
 C. CHECK FOR FULL LABEL
 NORMALIZE $\hat{x} = x / \sum x_i$
 VAR. LAB. $x_i =$
 CON. LAB. $A_i x_i =$

COMPLEXITY CLASSES

PLS: EXPTIME WITH BEST RESPONSE T-FNP: COMPUTE A SOL, KNOWING IT EXISTS, $f(x,y) = \text{TRUE!}$
 PPAD: EXPTIME WITH PATH TRAVERCAL

POTENTIAL GAMES

FUNCTION: $\phi: \phi(P) - \phi(P') = P_x(P) - P_x(\tilde{P}_x, P_{-x})$

PNE THM: ANY POTENTIAL GAME HAS AT LEAST ONE PNE
 BR-D ALWAYS CONVERGE!

SELFISH ROUTING: AGENT $\sum_{e \in P_x} p_e(x_e)$ WELFARE $\sum_x \sum_{e \in P_x} p_e(x_e)$

POTENTIAL $\phi_P = \sum_{e \in E} \sum_{k=1}^{x_e(P)} p_e(k)$ PoS=2 PoA=5/2

EQ.COND $\sum_{e \in P_x} p_e(x_e) \leq \sum_{\hat{P}_x \wedge P_x} p_e(x_e) + \sum_{\hat{P}_x \wedge P_x} p_e(x_{e+1}) \forall x, P$

CONGEST. GAMES: GENERALIZE THE ABOVE. E-RESOURCE

PoA: $(W(\text{WORST NE}) / W(\text{OPT}))^{\pm 1}$ PoS: $(W(\text{BEST NE}) / W(\text{OPT}))^{\pm 1}$ MIN/M
 USE SMOOTH ARGUMENTS. COMPARE ϕ AND WELF.

SMOOTH MIN $\exists h > 0, \mu < 1$:
 $(1-\mu)C(s) \leq C(s^*)h$
 PoA = $h / (1-\mu)$

SMOOTH MAX $\exists h > 0, \mu < 1$:
 $(1+\mu)C(s) \geq h \cdot C(s)$
 PoA = $1+\mu/\mu$

UTILITY GAMES: A. $V(s)$ SUBMODULAR: $V_x(s) + V_x(t) \geq V_x(s \cup t) + V_x(s \cap t) \forall s, t \subseteq C$
 B. $\sum_x V_x \leq V(s)$ C. $V_x \geq V(s) - V(s \setminus \{x\})$ IF V MONOTONE: $W(\text{PNE}) \geq 1/2$

MATCHINGS

TTC ALGO: $\Theta(m^2)$ MOST PREF: EDGE! FIND LOOPS AND REMOVE
 A. DSIC B. CORE HAS AN UNIQUE SOLUTION

THE CORE: $\{S: \text{NO COALITION CAN PARETO IMPROVE}\}$ PI: ALL =, AT LEAST ONE

GALE-SHAP: DEFERRED ACCEPTANCE, $\Theta(m^2)$ STRAT. PROOF FOR BOYS
 OPTIMAL MATCHING BOY BEST GIRLS WORST STABLE PART.

LP: $\sum_w X_{wm} = 1 \wedge \sum_m X_{wm} = 1 \wedge X_{wm} \geq 0 \forall m, w$ TUNIMODULAR
 $\sum_{j \in m} W X_{mj} + \sum_{\tilde{x} \in m} X_{\tilde{x}w} + X_{mw} \leq 1 \forall m, w$

THE CORE: FOR EACH COALITION S , $V(S) \leq V(I)$ WITH SET
 LP: $\min \sum_{\tilde{x}} X_{\tilde{x}}$ S.T. $\sum X_{\tilde{x}; \tilde{x} \in S} \geq V(S) \forall S \subseteq I, X_{\tilde{x}} \geq 0 \forall \tilde{x}$
 DUAL: $\max \sum_S V(S) \cdot Y_S$ S.T. $\sum_{S; \tilde{x} \in S} Y_S \leq 1 \forall \tilde{x}, Y_S \geq 0 \forall S$
 BALANCED: STRONG-DUALITY $\sum_{\tilde{x}} X_{\tilde{x}} = \sum_S V(S) \cdot Y_S = V(I)$

MATCHING MARKETS: n AGENTS WITH v_{ij} VALUATIONS FOR j ITEMS.

EQ: PRICES FOR A STABLE MATCHING

MATCHING ALGO: A. FIND UNSPLITABLE $x \in m, j$ HALVS: $|T(x)| \geq |x|$ LOOP UNTIL
 B. INCREASE PRICES OF $\pi(x)$ REDUCE SO THAT $\pi(x) = 0$ A MATCHING.
 POTENTIAL FN: $\phi(P) = \sum_{\tilde{x}} u_{\tilde{x}}(P) + \sum_j P_j$ WITH $u_{\tilde{x}} = \max_j v_{ij} - P_j$

DEFINITION: n AGENTS m GOODS WITH AGNOSTIC PRICES BUDGET \tilde{x} : $P \tau a_{\tilde{x}}$
 $q_{\tilde{x}}$ BUNDLE OF \tilde{x} EQ: SUP = DEM AND NO LEFT BUDGET
 1ST WAL. THM: AN EQ. P IS PARETO EFFICIENT $M_{\tilde{x}}(x+y) \geq M_{\tilde{x}}(x) + M_{\tilde{x}}(y)$
 2ND WAL. THM: ANY P.EFF SOL HAS SUPPORTING P
 EQ. EXISTENCE: CONTINUOUS $M_{\tilde{x}}; (2)$ MONOTONIC $M_{\tilde{x}}; (3)$ CONCAVE $M_{\tilde{x}}$
 PPAD-COMplete!

FISHER MKT: LINEAR $V_{\tilde{x}}$ AND BUYER/SELLER + MONEY \$\$\$

EIS. GALE: $\max \sum_{\tilde{x}} b_{\tilde{x}} \log(v_{\tilde{x}})$ S.T. $\sum_j v_{ij} x_{ij} = v_{\tilde{x}} \forall \tilde{x}; \sum_{\tilde{x}} x_{\tilde{x}} \leq 1; \forall j$

FLOW ALGO: $J, \pi(j): P_1 \cdot q(j) = b(\pi(j))$ TIGHT SET
 A. FIND TIGHT SET J : AGENT BUY AT PRICE P LOOP
 B. REMOVE ITEMS, AGENTS
 CHECK IF EQ: $J \rightarrow$ ITEMS: $P_i q_i$ AGENTS \rightarrow DT: $b_i s - t$ CUT

TRANSFERABLE UTILITIES

WALRASIAN MARKETS:

COMBINATORIAL MARKETS

PRIMAL: $\sum_{i \in I} \sum_{s \in S} x_{is} \cdot v_i(s)$

S.T. $\sum_{i \in I} \sum_{s \in S} x_{is} \leq 1 \quad \forall j \in S$

$\sum_{s \in S} x_{is} \leq 1 \quad \forall i$

DUAL: $\min \sum_i m_i + \sum_j p_j$

S.T. $m_i + \sum_{s \in S} p_s \geq v_i(s) \quad \forall i, s \in S$

TV. RELAX: SUPPORTING PRICES AND EQ! EG: WATERFRONT, HIS ON TREES
ELLIPSOID: SEPARATE BUNDLE CONSTRAINTS

IND. SET: \mathcal{E} ARE ITEMS \forall AGENTS WEIGHTED: N BUNDLES IFF SING. MINDED!

BEST. RESPON. DYNAMICS

REGRET: PAYOFF OF S - PAYOFF OF "BEST" FIXED ACTION

MAX GAIN: $C_i(s) - C_i(\tilde{s}_i; s_{-i}) > \epsilon C_i(s)$ BOUNDED JUMP: $B_m / \epsilon \cdot \ln(\phi p) B \geq 1$

LEMMA 1: $C_j(p) \geq \phi(p) / m$

$p_e(x+1) \leq B p_e(x)$ CONVERGENCE

LEMMA 2: $C_i(p) - C_i(\tilde{s}_i; p_{-i}) \geq \epsilon / B \cdot C_j(p)$ J BEST AGENT

POA THM: IF $\phi(s) \leq c(s)$ AND $(k-m)$ SMOOTH THEN $(2k)/(1-k)$ POA (FOR MIN.)

NO-REGR: A ACTIONS $P^t(a)$ CT ADVERSARIAL COST

MULT. WEIGHTS (MIN) $w^1(a) = 1, \forall a$

$P^t(a) = w^t(a) / W$

LEARN $w^{t+1} = w^t(a) \cdot (1 - \gamma C^t(a))$ $+1$ FOR MAX

THM: AFTER $T \geq 4 \ln(m)$ THEN AT MOST $2 \sqrt{\ln m / T}$ REGRET

COROLLARY: $e^{-\gamma \text{OPT}} - \gamma^2 T \leq W^{T+1} \leq m e^{-\gamma \sum_t \text{cost}(t)}$

CONVERGES TO COARSE (ϵ APPROX)

SPONSORED SEARCH AUCT.

DEFINITION: m BIDDERS WITH v_i VALUE PER CLICK / m SLOTS WITH α_j IMPR.

THM: MAX SW INDUCES AN ASSORTATIVE MATCHING (1^{TH} BID $\rightarrow 1^{\text{TH}}$ SLOT)

VCG-PRICES: $P_i = \sum_{p=i+1}^m v_p(\alpha_{p-1} - \alpha_p)$ NE + INDIV. RATIONAL!

GSP-PRICES: $b_1 = 1; b_i = P_{i-1} / \alpha_{i-1} \quad \forall i \geq 2$ POS = 1!