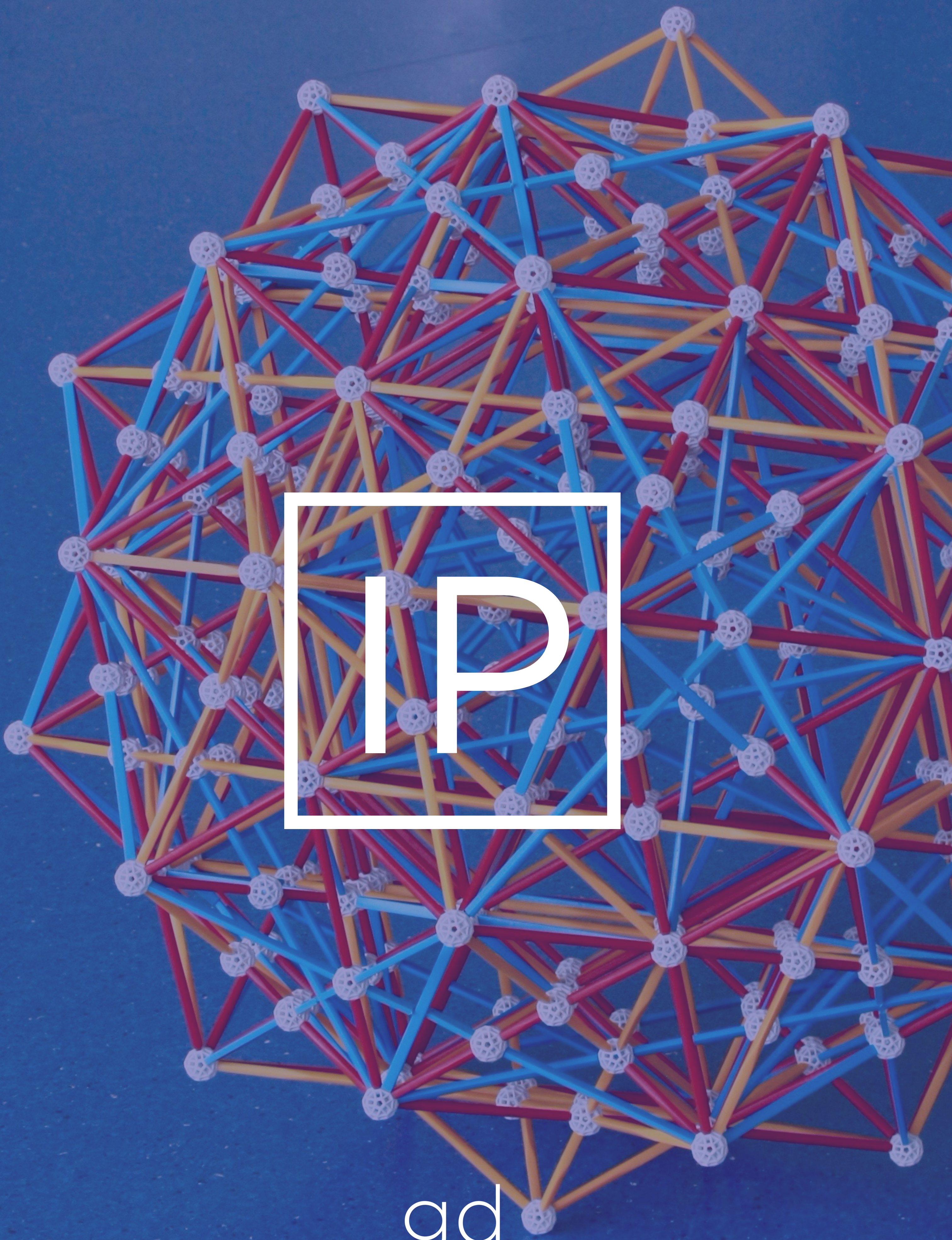

IP

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THE DREAMY WORLD OF INTEGER PROGRAMMING

A VERY SHORT REVIEW BY ME

CHAPTER ONE

IP FORMULATION

MIN $f(x)$

S.T. $x \in P$

$f(x)$ OBJECTIVE FUNCTION

x DECISION VARIABLES

P POLYTOPE OF AD. SOL.

INTEGER LINEAR PROGRAM

MIN $c^T x$

S.T. $Ax \leq b$

$x \in \mathbb{Z}^n, A \in \mathbb{R}^{n \times m}$

$, b \in \mathbb{R}^n$

BASIC HYPOTESIS

$\forall c_i \in C, a_{ij} \in A \Rightarrow$

$c_i \in \mathbb{Z}, A_{ij} \in \mathbb{Z}$

NOT RESTRICTIVE ONE CAN ALWAYS MULTIPLY :)

MODELISATION TRICKS

DISCRETE DOMAINS

$x \in \{1.5, 2, 2.5\}$

A. $x - 0.5y_2 - y_3 = 1.5$

$y_2 + y_3 \leq 1 \quad y_2, y_3 \in \{0,1\}$

B. $y \in \mathbb{N} \quad 3 \leq y \leq 5 \quad 2x - y = 0$

POLYTOPE AT-LEAST K

$A^i x \leq b + M^i(1-y_i) \quad \forall i=1,..q$

$\sum^q y_i \geq K \quad y_i \{0,1\}$

SUBTOUR ELIMINATION

$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 \quad \forall S \subset N, |S| \geq 2$

(LINEAR) RELAXATION

(P) MIN $z = c^T x$

S.T. $x \in X \subseteq \mathbb{R}^n$

(R) MIN $z_R = f(x)$

S.T. $x \in T \subseteq \mathbb{R}^n$

DEF R IS A RELAXATION FOR P IFF:

a. $X \subseteq T$

b. $f(x) \leq c(x) = c^T x$

LINEAR RELAXATION DROP THE INTEGRALITY CONSTRAINT

Rs GIVE VERY NICE GOSSIPS ABOUT

BOUNDS!

DUALITY

P MIN $c^T x$

S.T. $Ax = b \quad x \geq 0$

D MAX $y^T b$

S.T. $y^T A \leq c^T \quad y \in \mathbb{R}^m$

WEAK DUALITY $c^T x \leq y^T b$

OPTIMAL $c^T x = y^T b$

STRONG DUALITY OPTIMAL VALUES COINCIDE

COMPL. SLACKNESS

1. SATURATED DUAL

$\forall i: x_i > 0 \Rightarrow y^T a_i = c_i$

2. DUAL COMPLEMENTARITY

$\forall i: y^T a_i < c_i \Rightarrow x_i = 0$

3. SATURATED PRIMAL

$\forall j: y_j > 0 \Rightarrow a_j x = b_j$

4. PRIMAL COMPLEMENTARITY

$\forall j: a_j x < b_j \Rightarrow y_j = 0$

CHAPTER TWO

CONVEX COMBINATION

COMB $X = \sum_{i}^{K} \lambda_i v_i$ X IS A CONVEX COMBINATION OF POINTS $\{1, \dots, K\}$

CONVEX HULL: SET OF ALL CONVEX COMBINATIONS OF POINTS IN A SET

$$CONV(X) = \left\{ X = \sum_{i}^{K} \lambda_i v_i \mid \sum_{i}^{K} \lambda_i = 1, \lambda_i \geq 0, \forall i = 1, \dots, K \text{ AND } \{v_1, \dots, v_K\} \subseteq X \right\}$$

ALL EXTREMES BELONG TO THE SET

IDEAL FORMULATION: IS AN IP (ILP) WHERE THE POLITOPE $P = CONV(X)$
JUST SOLVE THE RELAX FOR THE OPTIMAL!

RELAXATION SOLUTION

IF THE R HAS AN OPTIMAL SOLUTION, THEN THERE IS AN OPTIMAL INTEGER FOR THE ORIGINAL PROBLEM

LEMMA $B^{m \times m}$ NON SINGULAR $\in \mathbb{Z}^{m \times m}$

$b \in \mathbb{Z}^m$

$B^{-1} b \in \mathbb{Z}^m \ \forall b$

$\Leftrightarrow DET(B) = \pm 1$

PROOF $B^{-1} = 1/_{DET(B)} \cdot ADJ(B)$

$DET(B) \in \mathbb{Z}^m \Rightarrow B^{-1} \in \mathbb{Z}^m$

$B^{-1} I = B^{-1} \in \mathbb{Z}^{m \times m}$

$\Rightarrow DET(B) DET(B^{-1}) = 1$

$\Rightarrow DET(B) = DET(B^{-1}) = \pm 1$

UNIMODUL

A MATRIX $B^{m \times m}$ WITH FULL RANK IS UNIMODULAR IF $DET(B) = \pm 1$ $\forall$ B BASE IN A

TOTAL UNIMODULARITY

A IS TOTALLY UNIMODULAR IF ALL ITS SUBMATRICES HAVE $DET(S) = \pm 1 \vee 0$

TU => UNIMODULARITY

TU FOR DUMMIES :

A. $a_{ij} = \{-1; 0; 1\} \ \forall a_{ij} \in A$

B. $\sum_{i}^{m} |a_{ij}| \leq 2 \ \forall j$ MAX 2 ON COLUMNS

C. $I = \underline{\underline{M_1}} \cup M_2$:

$\sum_{i \in M_1} a_{ij} = \sum_{i \in M_2} a_{ij} \ \forall a_{ij} \in A$

FOR ALL COLUMNS $\sum_{i}^{m} a_{ij} = 2$

=> A IS TOTALLY UNIMODULAR

// * IT CAN BE EMPTY

BOUND FOR KP0/1

$$X_i = \begin{cases} 1 & i < p \\ 0 & i > p \\ \dfrac{b - \sum_{i}^{p-1} a_i}{a_p} & i = p \end{cases}$$

BOUNDED SIMPLEX

$B = (a_p) \quad B^{-1} = 1/a_p \quad \overline{r_i} = c_i - \frac{c_p}{a_p} \cdot c_i$

$i < p \quad \frac{c_i}{a_i} \geq \frac{c_p}{a_p} \Rightarrow r_i \geq 0$ OPTIMAL

$i > p \quad \frac{c_i}{a_i} \leq \frac{c_p}{a_p} \Rightarrow r_i \leq 0$ THIS IS NOT FUNNY :(

CHAPTER THREE

SEPARATION — A SET OF DISJOINT SUBSETS SO THAT

$$\bigcup_{K \in K} S_K = S \qquad S_K \cap S_{K'} = \emptyset \ \forall K, K' \in K, \ K \neq K'$$

BRANCH & BOUND — (c) DOIG AND LAND - 1960

// IMPLICIT ENUMERATION ALGORITHM LEVERAGING ON BRANCHING INTO SUBPROBLEMS AND EVAULTING THEIR BOUNDS //

$\overline{Z}_K$ UB FOR SUBPROBLEM K. FEAS SOL!

$\underline{Z}_K$ LB FOR THE PROBLEM K RELAX

THE HOLY GRAIL

$$\overline{Z} = \underline{Z}$$

$$\overline{Z} = \min_K \overline{Z}_K \qquad \underline{Z} = \min_K \underline{Z}_K$$

NO BRANCHING IF:

- NO NODES $S_K = \emptyset$
- INTEGERS $\overline{Z}_K = \underline{Z}_K$
- BAD BOUND $\underline{Z}_K \geq \overline{Z}$

BRANCHING ON — VARIABLES, CONSTRAINTS, ECOTERIC COMBINATIONS OF VARIOUS STUFF

B&B V01

INIT $L = \emptyset$; $Z^* = \infty$; $K = 0$

TEST IF $L = \emptyset$: STOP!
IF $Z < \infty$ Z^* IS OPTIMAL
ELSE UNFEASIBLE

SELEC. PICK $P \in L$; $L = L \setminus \{P\}$

COMP RESOLVE RELAX IF $Z_P = -\infty$ UNBOUND
IF NOSOL GOTO TEST
IF ($\underline{X}_P$ FOUND)
INTEGER $\overline{X}_P = \underline{X}_P$ ELSE SEARCH FOR INT. SOL

IF ($\bar{X}_p$) FOUND AND ($\bar{z}_p < z^*$)
# NEW OPTIMAL
$X^* = \bar{X}_p$; $z = \bar{z}_p$;
$L = L \setminus \{q : \underline{z}_{p(q)} \geq z^*\}$ CLOSE BAD BOUNDED

IF $\underline{z}_p \geq z^*$ GOTO TEST

SEP SEPARATE SOMEHOW THE DOMAIN

SEPARATION RULES

A. FRACTIONAL VALUES

$x_i \leq \lfloor \underline{x}_i \rfloor \qquad x_i \geq \lceil \underline{x}_i \rceil$

B. CLOSEST TO INT

$i \in \underset{j \in F}{\mathrm{AMIN}} \, |f_i - 0.5|$
+ BOUND
+ EQ. TREE
+ BEST FIRST

C. "FURTHER" TO INT

$i \in \underset{j \in F}{\mathrm{AMAX}} \, |f_i - 0.5|$
- BOUND
+ FEASIBLE WITH DEPTH FIRST
- DEQ. TREE

PSEUDO COSTS BRANCH BASING ON THE x_i B/C IMPACT ON THE BOUND

P_i^+ FOR BRANCH $x_i \geq \lceil x_i \rceil$ K+1

P_i^- FOR BRANCH $x_i \leq \lfloor x_i \rfloor$ K+2

$$P_{ik}^+ = \frac{\underline{z}_{k+1} - \underline{z}_k}{1 - f_i} \qquad P_{ik}^- = \frac{\underline{z}_{k+2} - \underline{z}_k}{f_i}$$

$$P_i^+ = \sum_{k \in K_i} \frac{P_{ik}^+}{|K_i|} \qquad P_i^- = \sum_{k \in K_i} \frac{P_{ik}^-}{|K_i|}$$

BONES Δ ESTIMATORS

$D_i^+ = P_i^+ (1 - f_i)$ ↑ UP

$D_i^- = P_i^- f_i$ ↓ DOWN

EQ. TREE $\quad i = \mathrm{ARG} \underset{j \in F}{\mathrm{MAX}} \, \mathrm{MIN} \{D_j^+, D_j^-\}$

DEQ. TREE $\quad i = \mathrm{ARG} \underset{j \in F}{\mathrm{MIN}} \, \mathrm{MIN} \{D_j^+, D_j^-\}$

STRONG BRANCHING

BASING ON D_J^+ AND D_J^-

$$i = \text{ARG} \max_{J \in C} \min \{D_J^+, D_J^-\}$$

INCREASES THE MOST THE SMALLEEEEEEEST BOUND

GUB e SOS

GENERALIZED UPPER BOUND

$$\sum_{J \in S} X_J \leq U$$

$$\sum_{J \in S} X_J \lessgtr \left\lceil \sum_{J \in S} X_J \right\rceil$$

TRICK $Y = \sum_{J \in S} X_J$ AND BRANCH ON J

$0 \leq Y_S \leq \lfloor Y_S \rfloor$ $\lceil Y_S \rceil \leq Y_S \leq U$

SPECIAL ORDERED SETS

$$\sum_{J \in S} X_J = 1$$

ASSUME $X_J \geq X_{J+1}$ $J = \{1, \ldots, P-1\}$

$$r = \text{ARG} \min_{q \in \{1, P-1\}} \left| \sum_{i}^{q} X_J - 0{,}5 \right|$$

$\{1, r\} = S_1$ $S_2 = \{r+1, P\}$

$\sum_{J \in S_1} X_i = 1$ $\sum_{J \in S_2} X_i = 1$

$X_J = 0 \ \forall J \in S_2$ $X_J = 0 \ \forall J \in S_1$

EXPLORATION STRATEGIES

DEPTH-FIRST

DEEPEST NODE IN TREE

PROS
1. FEASIBLE SOL
2. REOPTIM
3. L ≤ DEPTH

CONS "WALK ON THE WILD SIDE"

COMPOSE YOUR OWN MIX!

MIN

BEST-FIRST

SMALLEST LOWER BOUND

PROS
1. CHANGE YOUR MIND
2. MIN(EXPLORED)
3. BEST LB

CONS
1. L IS HUUUGE
2. REOPTIMIZATION

BEST-PROJ
SMALLEST

$$E_p = \underline{z}_p + \left(\frac{\bar{z} - \underline{z}_0}{g^0} \right) g^p$$

F^p SET OF FRACT. VARS

$g^p = \sum_{i \in F^p} \min\{f_i, 1 - f_i\}$ SUM OF INFEASIBILITIES

BEST-ESTIM
SMALLEST

$$E_p = \underline{z}_p + \sum_{i \in F^p} \min\{P_i^- f_i, P_i^+ (1 - f_i)\}$$

DOES NOT REQUIRE $\bar{z}$

PRE-PROCESSING

REFORMULATE THE CONSTRAINTS
/SIMILAR TO CP PROPAGATION!

REDUCED COSTS FIXING

FIX VARIABLES BASING ON KNOWN BOUND AND DUAL

$$r = c - A^T \lambda$$

$$\bar{r}_i > \bar{z}_{INC} - z_{REL}$$

$$\bar{r}_i > \bar{z} - \lambda^T b$$

PROOF $\hat{x}$ SOLUTION (FEAS) WITH $\hat{z}$

$$\begin{aligned}
\hat{z} &= c^T \hat{x} - \lambda^T b + \lambda^T b \\
&= c^T \hat{x} - \lambda^T (A \hat{x}) + \lambda^T b \\
&= \hat{x} (c^T - \lambda^T A) + \lambda^T b \\
&= \bar{r}^T \hat{x} + \lambda^T b \quad \text{FEASIBLE!}
\end{aligned}$$

$$\hat{z} = \sum_{i}^{n} \bar{r}_i x_i + \lambda^T b \geq \bar{r}_j \hat{x}_j + \lambda^T b$$

$\hat{x} \geq 0$ & $\bar{r} \geq 0$

$$\geq \bar{r}_j + \lambda^T b > \bar{z} - \lambda^T b + \lambda^T b = \bar{z}$$

CONTRADICTION

CHAPTER FOUR

DEFINITIONS

POLYHEDRON

SUBSET OF $\mathbb{R}^m$ DEFINED BY A FINITE SET OF INEQUALITIES

$$P_e = \{x \in \mathbb{R}^m : Ax \leq b\}$$

POLYTOPE

A BOUNDED POLYHERRON

$$\exists M \in \mathbb{R} : \|x\| \leq M \quad \forall x \in P$$

ALWAYS CONVEX

PROOF $P = \{X \in \mathbb{R}^m : Ax \leq b\}$

$X \in \mathrm{CONV}(P) \Rightarrow \exists k \in \mathbb{Z}, \lambda \in \mathbb{R}^k_+, V_i \in P : \sum_k \lambda_i V_i = x \wedge \sum_k \lambda_i = 1$

$$Ax = A \sum^k \lambda_i V_i \leq \sum_k \lambda_i b = b$$

$$\Rightarrow x \in P \wedge \mathrm{CONV}(P) \in P$$

MINIMIZAT (MAX)

IF $X \subseteq \mathbb{R}^m$ IS COMPACT THEN $Z_P = \min_X C^T x = Z_C = \min_{x \in \mathrm{CONV}(x)} C^T x$

PROOF Z_C IS CONVEX-HULL, AND Z_P A RELAXATION

$$\Rightarrow Z_P \leq Z_C$$

$\mathrm{CONV}(X) = \emptyset \quad Z_C = Z_P = \infty$

$\mathrm{CONV}(X) \neq \emptyset$

$$X_C = \sum_k \lambda_i V_i \wedge \sum_k \lambda_i = 1$$

SMALLEST COST J $\quad C^T V_J \leq C^T V_i \; \forall i$

$$Z_C = C^T \cdot \left(\sum_k \lambda_i V_i\right) \geq \sum_k \lambda_i V_J C^T = C^T \cdot V_J \sum_k \lambda_i = C^T V_J \geq Z^P$$

SEPARATION HYPERPLAN

GIVEN A SET $S = \{V_1, V_2, \dots V_m\} \in \mathbb{R}^m$ AND A POINT $V_0 \in S \setminus \mathrm{CONV}(S)$, IT EXISTS A SEPARATION HYPERPLAN BETWEEN THE TWO.

$$\pi^T x = \pi_0 : \quad \pi^T x \leq \pi_0 \;\; \forall x \in \mathrm{CONV}(S), \; \pi^T V_0 > \pi_0$$

SEPARATION PROBLEM

PRIMAL FORM

$$\min_{\lambda} \sum_{K} 0 \cdot \lambda_i \quad \text{INFEASIBLE}$$

$$\sum_{K} \lambda_i V_i = V_0 \quad (y \in \mathbb{R}^n)$$

$$\sum_{K} \lambda_i = 1 \quad (y_0 \in \mathbb{R})$$

$$\lambda \geq 0$$

DUAL FORM

$$\max_{y_0, y} V_0^T y + y_0 \quad \text{UNBOUNDED}$$

$$V_i^T y + y_0 \leq 0 \quad \forall i \in K$$

DUAL SOLUTION

$$V_0^T \bar{y} + y_0 \geq 0 \quad \pi_0 = -y_0 \quad \pi = \bar{y}$$

$$V_i^T y + y_0 \leq 0$$

VALID INEQUALITY

"GIVEN A SET $S \subset \mathbb{R}^n$, $\pi^T x \leq \pi_0 \wedge \pi \neq \bar{0}$ IS A VALID INEQUALITY IF $S \subseteq \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\}$"

INTEGER SET $\quad \pi^T x \leq \pi_0 \quad \forall x \in S$

REDUNDANCY (1-TO-1)

$\pi^T x \leq \pi_0$ DOMINATES $\mu^T x \leq \mu_0$ IF.

A. BOTH ARE VALID FOR $S \subset \mathbb{R}^n_+$.

B. $\exists K > 0 : \pi \geq K\mu \wedge \pi_0 \leq K\mu_0$ WITH $(\pi, \pi_0) \neq (\mu, \mu_0)$

FOLLOWS $S^\pi \subseteq S^\mu$

REDUNDANCY (1-TO-m)

$\pi^T x \leq \pi_0$ IS REDUNDANT FOR m INEQUALITIES IN THE FORM $\mu_i^T x \leq \mu_{0,i}$ IF

A. ALL INEQUALITIES ARE VALID FOR $S \subset \mathbb{R}^n_+$

B. $\exists \bar{K} \in \mathbb{R}^m : \sum_i (K_i \mu_i)^T x \leq \sum_i K_i \mu_{0,i}$ DOMINATES $\pi^T x \leq \pi_0$

FACE

"A FACE OF P IS THE SET $F = \{x \in P : A^0 x = b_0\}$"

DIM(F)		
	0	VERTEX
	$1 \cdots m$	m-FACE
	$\dim(P) - 1$	FACET! NECESSARY TO DESCRIBE P

EXTREME POINT

" u IS AN EXTREME POINT FOR P IF THERE EXISTS A VALID INEQUALITY $\pi^T x \le \pi_0$ SO THAT ITS FACE CONTAINS ONLY u "

SET $EXT(P)$

CONV(P) CAN BE DESCRIBED AS CONVEX COMBINATION OF EXTREME POINTS

EXTREME RAY

" $r \in \mathbb{R}^n$ IS AN EXTREME RAY OF P IF IT IS A RAY OF P AND $\nexists\, r_1, r_2$ RAYS OF P: $r_1 \neq \lambda r_2 \ \forall \lambda \ge 0 \wedge r = \frac{1}{2} r_1 + \frac{1}{2} r_2$ "

RAY A VECTOR r IS A RAY OF P IF $x + \alpha r \in P \ \forall x \in P, \alpha \ge 0$

MINKOVSKI THEOREM

" A CONVEX SET CAN BE DEFINED BY THE LINEAR COMBINATION OF ITS EXT AND EXT_R "

$$P = Q := \left\{ \sum_K \lambda_i v_i + \sum_p \alpha_j r_j \;\middle|\; \sum_K \lambda_i = 1, \lambda \in \mathbb{R}^K_+, \alpha \in \mathbb{R}^p_+ \right\}$$

AFFINELY INDEP.

" THE SET OF POINTS $\{v_1, v_2, \dots, v_m\}$ IS AFFINELY INDEPENDENT IF $\{v_2 - v_1, v_3 - v_1, \dots, v_m - v_1\}$ IS LINEARLY INDEPENDENT "

LINEAR INDEPENDENT $\sum_K \alpha_i v_i = 0 \Rightarrow \sum_K \alpha_i = 0$

DIMENSION

" THE DIMENSION $DIM(P)$ OF A POLYHEDRON IS GIVEN BY

$$DIM(P) : \{ AFF\_INDEP\_POINTS - 1 \}$$

$$DIM(P) : m - RANK(A_{=}^{n \times m}) \text{ SAT. AT } =$$

SMALLEST SUBSPACE CONTAINING ALL THE POINTS $x_1 - x_0$, $x_0, x_1 \in P$

CHAPTER FIVE

CHVÁTAL GOMORY

"GIVEN A POLYTOPE $S = \{x \in \mathbb{R}^n ; Ax \leq b\}$ WITH m INEQUALITIES $A_i^T x \leq b$ $\forall i = 1, \dots m$, THE CG INEQUALITY IS GIVEN BY

$$\left\lfloor \sum_{i=1}^{m} u_i A_i^T \right\rfloor x \leq \left\lfloor \sum_{i=1}^{m} u_i b_i \right\rfloor \quad \forall u \in \mathbb{R}_m^+$$

LINEAR COMBINATION — OF VALID INEQUALITIES (WEIGTHED AND SUMMED UP) IS STILL VALID

RANK 1 — FOR ALL C-G INEQUALITY FROM $Ax \leq b$ SET X'

RANK K — BETWEEN $Ax \leq b$ AND CG OF RANK $(K-1)$ SET X^K

THEOREM — IF P IS A RATIONAL (INT) POLITOPE, THEN THERE EXIST A K:

$$X^K = \mathrm{CONV}(P \cap \mathbb{Z}^n)$$

K — CHVATAL RANK

GOMORY'S CUTTING ALGORITHM

PROBLEM $\{\min c^T x ;\ Ax = b,\ x \in \mathbb{Z}^n\}$

OPTIMAL RELAX — B BASE AND D NON-BASIC

$$a_{ij} = (B^{-1}A)_{ij} \quad b_i = (B^{-1}b)_i$$

AT OPTIMALLITY
$$C_j \geq 0 \quad \forall j \in B \cup D$$
$$X_i = b_i \quad \forall i \in B$$
$$X_i = 0 \quad \forall i \in D$$

$$\Rightarrow x_i + \sum_{j \in D} a_{ij} x_j = b_i$$

$$\begin{cases} x_i + \sum_{j \in D} \lfloor a_{ij} \rfloor x_j - \lfloor b_i \rfloor = f_i - \sum_{j \in D} f_{ij} x_j \\ f_i = b_i - \lfloor b_i \rfloor > 0 \qquad \forall j \in D \\ f_{ij} = a_{ij} - \lfloor a_{ij} \rfloor \end{cases}$$

$$\Rightarrow f_i - \sum_{j \in D} f_{ij} x_j \leq 0$$

THE GOMORY CUT

$$\sum_{j \in D} f_{ij} x_j \geq f_i$$

ALSO PREVENT FURTHER DUMMY BASES!

DISJUNCTIVE INEQUALITIES

" CONSTRAIN THE POLYTOPE WITH A SET OF CONSTRAINTS LINKED BY AN OR "

$S_1, S_2 \subseteq \mathbb{R}^n_+ : \; S_1 \cap S_2 = \emptyset, \; S_1 \cup S_2 = S$

PROPOSITION $\sum_{j=1}^{n} \pi_j^i x_j \leq \pi_0^i \quad \forall i = 1,2 \text{ FOR } S_i$

THEN THE FOLLOWING INEQUALITY IS VALID FOR S

$$\sum_{j=1}^{n} \min\{\pi_j^1, \pi_j^2\} x_j \leq \max\{\pi_0^1, \pi_0^2\}$$

PROPOSITION ILP ON $S = P \cap \mathbb{N}^n$

$P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$

$P_1 = P \cap \{x \in \mathbb{R}^n_+ : \alpha^T x \leq \alpha_0\}$

$P_2 = P \cap \{x \in \mathbb{R}^n_+ : \alpha^T x \geq \alpha_0 + 1\}$

$\pi^T x \leq \pi_0$ IS VALID FOR $\text{CONV}(P_1 \cup P_2)$ IF:

$\exists \; \bar{u}_i \geq 0, \; v_i \geq 0 \quad i = \{1,2\}$:

A. $\pi^T \leq u_1^T A^T + v_1 \alpha^T$

B. $\pi^T \leq u_2^T A^T - v_2 \alpha^T$

C. $\pi_0 \geq u_1^T b + v_1 \alpha_0$

D. $\pi_0 \geq u_2^T b - v_2 (\alpha_0 + 1)$

CLIQUE INEQUALITIES

" GIVEN A CONSTRAINT $\sum x_i \leq 1$, A CLIQUE IS A SET OF VARIABLES IN CONFLICT, HENCE EACH ONE MUTUALLY EXCLUDE THE OTHER "

SEPARATION PROBLEM $\sum_j \bar{x}_j s_j \; (\text{MAX}) \text{ ST: } s_i + s_j \leq 1 \;\; \forall i,j \in F$

$F = \{(i,j) \text{ NOT IN CONFLICT}\} \quad s \in \mathbb{B}^n$

CONFLICT GRAPH

V SET OF VARIABLES $\bar{x}_j \; \forall j$

E SET (u,v) IF u, v ARE IN CONFLICT

NP-HARD SORRY FOR THAT... :(

LIFTING A MINIMAL CLIQUE DEFINES A FACET IF IT IS OF MAXIMAL CARDINALITY. BY SEARCHING FOR A CLIQUE IN WICH ALL VS ALL, WE "LIFT" IT BY ADDING OTHER VARIABLES IN CONFLICT WITH SOME OTHERS

ODD-CYCLES INEQUALITIES

" GIVEN A SET OF $|J|$ VARIABLES J, SO THAT $|J|$ IS ODD AND ALL VARIABLES ARE IN CONFLICT, THE FOLLOWING INEQUALITIES ARE VALID

$$\sum_{j \in J} x_j \leq \frac{|J|-1}{2}$$

$|J|=3$ CLIQUE!

VIOLATED WHEN

$$\left(\sum_J x_j\right) \cdot 2 > |J|-1$$

$$\sum_{m=1}^{|J|-1} (1-y_m-y_{m+1}) + 1 - y_{|J|} - y_1 < 1$$

CONFLICT GRAPH

$V \quad (1-y_{m_1}-y_{m_2})$

$E \quad (u,v) = (m_1, m_2)$

CYCLE OF ODD CARDINALITY IN $G_R = (V,E)$

LIFTING

ADD VERTICES WITH COEFFICIENT GIVEN BY A SEP. PROBLEM

$$C_K = \frac{|J|-1}{2} - z_K$$

$$z_K = \max \sum_{i \in C} d_i w_i$$

$w_i + w_j \leq 1 \quad \forall (i,j) \in E(C \cup \{k\})$

$w_K = 1$

$w_i \in \mathbb{B}^n$

KNAPSACK 0/1 CUTS

COVER INEQUALITIES

$$\sum_{i \in C} x_i \leq |C|-1 \qquad C \subseteq S : \sum_{i \in C} a_i > b$$

MINIMAL

TAKING OUT ANY OF THE ELEMENTS IN C IMPLY DESTROYING THE COVER DEFINES A FACET OF CONV(S)

SEPARATION PROBLEM

$$\min_w \sum_{i \in N} (1-\hat{x}_i) w_i < 1$$

$a^T w \geq b+1$

$w \in \mathbb{B}^n$

$\hat{x}_i$ MUST BE FRACTIONAL OR ZERO

$\hat{x}_i = 0 \quad w_i = 0 \qquad \hat{x}_i = 1 \quad w_i = 1$

CHAPTER SIX

DANTZIG WOLFE DECOMPOS.

" ALGORITHM FOR SOLVING LINEAR PROBLEM WITH A BLOCKANGULAR STRUCTURE "

$$
\begin{matrix}
A_1 & A_2 & \cdots & A_m & b \\
D_1 & 0 & \cdots & 0 & e_1 \\
0 & D_2 & \cdots & 0 & e_2 \\
 & & \ddots & & \vdots \\
0 & 0 & \cdots & D_m & e_m
\end{matrix}
$$

COUPLING — "CONNECTING" OR "COMPLICATED" CONSTRAINT. KEPT IN MASTER PROBLEM

EASY — OR BLOCKANGULAR CONSTRAINT. OFTEN CONSTITUTE AN EASIER SUB PROBLEM

REMINDER: MINKOWSKIJ — $\forall x \in \mathrm{CONV}(P),\ \exists \lambda \geq 0,\ \theta_p \geq 0 : \sum_{p\in\Omega} \theta_p = 1,$

$$x = \sum_{p\in\Omega} \theta_p \cdot w_p + \sum_{r\in\Gamma} \lambda_r \rho_r$$

REFORMULATION

$$
\begin{cases}
c_p = C^T w_p & a_p = A w_p \\
c_r = C^T \rho_R & a_r = A \rho_r
\end{cases}
$$

$$\min_{x,\theta,R}\ \sum_{p\in\Omega} c_p \theta_p + \sum_{r\in\Gamma} c_r \lambda_r \qquad \Big(\sum_{k\in K}\Big)$$

$$\sum_{p\in\Omega_k} a_{pk}\theta_{pk} + \sum_{r\in\Gamma} a_{rk}\lambda_{rk} = b \qquad \Big(\sum_{k\in K}\Big)$$

$$\sum_{p\in\Omega_k} \theta_{pk} = 1 \qquad \forall k\in K$$

$$\theta_{pk} \geq 0 \qquad \forall p \in \Omega_{pk} \quad \forall k\in K$$

$$\lambda_{rk} \geq 0 \qquad \forall r \in \Gamma \quad \forall k \in K$$

$$x_k = \sum_{p\in\Omega_k} \theta_{pk} w_{pk} + \sum_{r\in\Gamma} \lambda_{rk} \rho_{rk} \in \mathbb{Z}(\mathbb{N}) \qquad \forall k\in K$$

DOMAIN — $\Delta V \Rightarrow V$ IN V

SOLUT. — INTEGER COMBINATIONS OF EXTREME POINTS OF ΔV

BRANCH AND PRICE

PRICING MASTER PROBLEM

CONTAINS COUPLING CONSTRAINTS AND CHECK FOR HIGHER LEVEL FEASABILITY (PM_i)

$$\min_{x,\theta} \sum_{p \in \Omega^i} c_p \cdot \theta_p$$

$$\sum_{p \in \Omega^i} a_p \cdot \theta_p = b \quad (\pi)$$

$$\sum_{p \in \Omega^i} \theta_p = 1 \quad (\sigma)$$

$$\theta_p \geq 0 \quad \forall p \in \Omega^i$$

$\downarrow \sigma, \pi$

$\uparrow \theta_i$

RESTRICTED — WORKS ON $\Omega_i \subset \Omega$

OPTIMALLITY — $c_j \geq 0 \;\forall\; \theta_j \in \Omega \setminus \Omega_i$

SUB PROBLEM

GENERATE COLUMNS FOR THE MASTER PROBLEM

$$c_p^i = c_p - (\pi^i)^T a_p - \sigma^i$$

$$\min_{x} \; (c^T - (\pi^i) A)\, x - \sigma^i$$

$$Dx = e$$

$$x \in \mathbb{N} \quad \text{INTEGRALITY}$$

OPTIMALLITY — $z \geq 0$ MEANS COLUMNS ARE OPTIMAL OTHERWISE NEW $\theta_i \Rightarrow PM_{i+1}$

A. CREATE Ω_i AND SOLVE $PM_1 = \theta_i, \pi, \sigma$

B. SOLVE SP_i WITH σ, π, Ω_i

IF $z_{SP_i} \geq 0$ OPTIMAL!

ELSE θ_{i+1} WITH c_p, a_p

$\Omega_{i+1} = \Omega_i \cup \{p\}$ AND REPEAT 2

"CONVERGES TO OPTIMALITY IN A FINITE NUM. OF ITERATIONS"

BRANCHING

DO NOT — BRANCH ON θ

BRANCH — ON VARIABLES OF ORIGINAL FORM, AND REWRITE THEM

BOUNDS

$$D_{DW}^{(INT)} \subseteq D_{DW}^{(CONT)} = D_{REL}^{(x)}$$

INT. PROP. SP — ALL EXTREME POINTS OF REL(SP) ARE INTEGER SAME BOUNDS AS REL(DW_PM)

CHAPTER SEVEN

LAGRANGIAN RELAXATION

RELAX CONSTRAINTS AND PENALIZE THEIR VIOLATION WITH SO-CALLED MULTIPLIERS

$$z^P = \min_x c^T x$$

$$A_1 x = b_1 \quad (\pi) \text{ UNBOUNDED}$$

BINDING CONSTRAINTS

$$A_2 x \geq b_2 \quad (\mu) \text{ BOUNDED } \geq 0$$

$$Dx = e$$

$$L(\pi, \mu) = \min_x \; c^T x - \pi^T (A_1 x - b_1) - \mu^T (A_2 x - b_2)$$

$$\text{S.T. } Dx = e$$

LAGRANGIAN OBJECTIVE — CAN BE REWRITTEN AS

$$L(\pi, \mu) = \pi^T b_1 + \mu^T b^2 + \min_x (c^T - \pi^T A_1 - \mu^T A_2) x$$

PIECEWISE LINEAR + CONCAVE

SEPARABLE! — EACH SUB PROBLEM WORKS ON A SUBSET

$$\Delta^K = \{ x^K : D_K x_K = e_K , \; x_K \in X^K \}$$

LOWER BOUND — ON THE ORIGINAL

$$L(\pi, \mu) \leq z^P$$

DUAL LAGRANGIAN

FINDS THE BIGGEST LOWER BOUND GIVEN BY ANY OF THE LAGRANGIAN

$$L^* = \left\{ \max_{\pi, \mu} L(\pi, \mu) \text{ S.T. } \mu \geq 0 \right\}$$

BOUNDS $\quad L(\pi, \mu) \leq L^* \leq z^P \leq c^T x$

SUBGRADIENT

APPLY IT FOR $J(\pi, \mu) = -L(\pi, \mu)$ SINCE IT IS NON-DIFFERENTIABLE

SUBGRADIENT — $f : \mathbb{R}^n \to \mathbb{R}$ CONVEX

$$\partial f(\bar{x}) = \{ \gamma \in \mathbb{R}^n : \forall x \in \mathbb{R}^n \; f(x) \geq f(\bar{x}) + \gamma^T (x - \bar{x}) \}$$

AT OPTIMALITY — GIVEN x^* OPTIMAL FOR SP

$$\gamma(x^*) = \begin{pmatrix} A_1 x^* - b_1 \\ A_2 x^* - b_2 \end{pmatrix}$$

GRADIENT UPDATE

$$(\pi_T, M_T) = [(\pi_{T-1}, M_{t-1}) - S_t \gamma_t (x^*)^T]^+$$

+ ALL VECTORS IN M SHOULD BE GREATER-EQUAL THAN 0

ROBBINS MONROE

ENSURE CONVERGENCE

$$\lim_{t \to \infty} s_t = 0 \qquad \lim_{t \to \infty} \sum_{j=1}^{t} s_t = \infty$$

EMPIRICALLY GOOOOOD :)

$$s_t = \alpha_t \frac{J(\pi_{t-1}, M_{t-1}) - \underline{z}}{\|\gamma_t\|^2}$$

$$\alpha_t \in [0, 2]$$

FINDING FEASIBLE SOLUTIONS

HEURISTICALLY RELAX A CONSTRAINT, ELIMINATE ONE OTHER AND FORMULATE A REPARATION PROBLEM FOR THE LATTER ONE

CHAPTER EIGHT

BENDER DECOMPOSITION

"ROW GENERATION" APPROACH, WHICH EXPLOITS THE PARTITIONING OF VARIABLES: IF FIX y THEN THE PROPLEM OR x BECOMES EASIER

$$\min \quad c^T x + d^T y$$
$$Ax + By = e$$
$$x \geq 0 \qquad y \in Y$$

DUAL BLOCK ANGULAR $\quad A_i x_i = e_i - B_i y$

$$\begin{matrix} A_1 & & & B_1 & e_1 \\ & A_2 & & B_2 & e_2 \\ & & \ddots & \vdots & \vdots \\ & & A_m & B_m & e_m \end{matrix}$$

PRIMAL PROBLEM $\quad \min_{y \in Y} \; h(y) + d^T y$

SUB PROBLEM $\quad \min_{x \in X} \; c^T x = h(y)$

$$Ax = e - By \quad (u \text{ DUALS})$$
$$x \geq 0$$

DUAL SUBPROBLEM

$$\max_{u} \; h(y) = u^T(e - By)$$
$$u^T A \leq c$$

FEASABILITY OF PRIMAL

"SP IS FEASIBLE FOR $y \in Y$ IF

$$u^T(e - By) \leq 0 \quad \forall u^T : u^T A \leq 0$$

NEW CONST. FOR THE PRIMAL RAY

$$\rho_r^T (e - By) \leq 0 \quad \forall r \in \Gamma$$

FEASABILITY OPTIMALITY CUTS

OPTIMALITY EXT. POINT $\quad z - w_p^T(e - By) \geq 0 \quad \forall p \in \Omega$

WITH $z = c^T x$ OF SUBPROBLEM

FEASABILITY RAYS $\quad \rho_R^T(e - By) \leq 0 \quad \forall r \in \Gamma$

MASTER PROB.

$$\min \; z + d^T y \qquad y \in Y$$
$$z - w_p^T(e - By) \geq 0 \quad \forall p \in \Omega$$
$$\rho_R^T(e - By) \leq 0 \quad \forall r \in \Gamma$$

SOLVING PROCESS

1. INITIALIZE Ω_1, Π_1 AND SOLVE MPR_1

2. IF NOT FEASIBLE: STOP. MPR
   ELSE GET Z_m, Y_m *MIGHT BE UNBOUNDED

   A. SOLVE SP WITH Y_m

      UNBOUNDED STOP. UNBOUNDED

      INFEASIBLE NEW FEAS. CUT Π

      FEASIBLE NEW OPTIMALITY CUT Ω

INTEGER PROGRAMMING CHEATSHEET

MODELING

DISCRETE SET $x \in \{2, 2.5, 1.5\} \Rightarrow x + 0.5y_1 + y_2 = 1.5 \quad y_1 + y_2 \le 1 \quad y_1, y_2 \in \mathbb{B}$

DISJOINT SETS $x_1 \le x_2 + M(1-y) \vee x_2 \le x_1 \ (My)$

LINEAR IF-THEN $\sum y_i = 1, \ x = \sum x_i, \quad y_i \cdot LB_i \le x_i \le UB_i \cdot y_i \quad c^T x = y_i k_i + x_i$

AT-LEAST K/Q $A^i x \le b_i + M(1-y_i) \quad \sum y_i \ge K$

IMPLICATIONS $x_1 \Rightarrow x_2 : x_2 \ge x_1 \quad x_1 \Rightarrow ! x_2 \quad x_1 \le 1 - x_2$

SUBTOUR ELIM. $\sum_{i \in S} \sum_{j \notin S} x_{ij} \ge 1 \quad \forall S \subseteq N, |S| \ge 2, |S| \le 2m-1$

SIMPLEX RECALL

$\lambda^T = c_B B^{-1} \quad x_B = B^{-1} b \quad r_D^T = c_d^T - \lambda^T D \quad z = c_B^T x_B^T \quad \le -SL \quad \ge +SL$

DUALITY $D = \{\max y^T b \, ; \, Ay \le c, \, y \in \mathbb{N}\}$ WEAK DUALITY $c^T x \le y^T b$

COMP. SLACKNESS A. $\forall i : A_i x_j < b_i \Rightarrow y_i = 0$

B. $\forall j : x_j = 1 \Rightarrow A_i y_i = c_j$

UNIMODULAR

A IS UNIMODULAR IF: FULL RANK + DET = ±1

TOTAL UNIMOD. EACH SUBMATRIX HAS DET $= \{0, \pm 1\}$ A. $\forall a_{ij} = \{0, \pm 1\}$ B. $\sum_{i=1}^{m} |a_{ij}| \le 2$

C. $I = M_1 \cup M_2 : \sum_{i \in M_1} a_{ij} = \sum_{i \in M_2} a_{ij}$ IF $\sum_i a_{ij} = 2$

BRANCH AND BOUND

$\bar{z} = \min_K \bar{z}_K \quad \underline{z} = \min_K \underline{z}_K$ FATHOMING $\cdot \, S_K = \emptyset \ \cdot \ \bar{z}_K = \underline{z}_K \ \cdot \ \underline{z}_K \ge \bar{z}^*$

BRANCHING

FRACTIONAL $x_1 \le \lfloor x_i^* \rfloor \quad x_1 \ge \lceil x_i^* \rceil$

CLOSEST INT $i \in \arg\min |f_i - 0.5|$ BOUND + EQ. TREE DEPTH FIRST

FAR FROM INT $i \in \arg\max |f_i - 0.5|$ BOUND + DEQ. TREE BEST FIRST

PSEUDO COSTS $P_{iK}^{+} = \frac{\underline{z}_{K+1} - \underline{z}_K}{1 - f_i} \quad P_{iK}^{-} = \frac{\underline{z}_{K+2} - \underline{z}_K}{f_i} \quad P_i^{+} = \sum_K P_{iK}^{+} / |K_i^{+}| \quad P_i^{-} = \sum_K P_{iK}^{-} / |K_i^{-}|$

NOT ADDITIVE! STRONG BRANCHING $D_i^{+} = P_i^{+}(1 - f_i) \uparrow \quad D_i^{-} = P_i^{-} f_i \downarrow$

EQ. TREE $i = \arg\max_{j \in F} \min\{D_j^{-}, D_j^{+}\}$ DEQ. TREE ARGMIN //

GUB AND SOS

GUB $\sum x_i \le UB = y \quad L \le \lfloor y \rfloor \quad \lceil y \rceil \le UB$

SOS $\sum x_i = 1 \quad r = \arg\min_{q} |\sum_{q} x_i - 0.5| \quad \sum_{i=1}^{r} x_i = 1 \vee \sum_{K+1}^{m} x_i = 1$

EXPLORATION

DEPTH-FIRST +FEAS +REOPT + L ≤ DEPTH -WILDSIDE

BEST-FIRST SMALLEST LB +BEST -HUGE L -REOPT

BEST-PROJ $E_p = z_p + \left(\frac{\bar{z} - \underline{z}_0}{g_0} \right) g_p \quad g_p = \sum_{i \in F} \min\{1 - f_i, f_i\}$

REQUIRES $\bar{z}$

BEST-ESTIM $E_p = \underline{z}_p + \sum_F \min\{P_i^{-} f_i, P_i^{+}(1 - f_i)\}$ DO NOT REQUIRE $\bar{z}$

r-COST FIX MIN $r_i > \bar{z}_{INC} - z_{REL}(\lambda^T b) \Rightarrow x_i = 0 \ (LB)$

$r_i < \underline{z}_{REL}(\lambda^T b) - \bar{z} \Rightarrow x_i = 1 \ (UB)$

POLYHEDRAL THEORY

POLYSTUFF

HEDRON FINITE NUMBER OF INEQUALITIES $P = \{x \in \mathbb{R}^n : Ax \le b\}$

MINKOWSKI $P = \{ \sum_i \lambda_i v_i + \sum_p \alpha_j r_j : \sum \lambda_i = 1, \ \lambda_i, \alpha_j \in \mathbb{R}^+ \}$

TOPE BOUNDED. $\exists M \in \mathbb{R} : \|x\| \le M \ \forall x \in P$ ALWAYS CONVEX

SEP. HYPERP $S = conv(X) = \{v_1, v_2, \dots v_m\} \quad \exists \bar{\pi}, \pi_0 : \pi^T x \le \pi_0 \ \forall x \in S, \ \pi^T b > \pi_0$

KNOWING EXTREME POINTS INFEAS $\{\max \sum_K 0 \lambda_K \, ; \, \sum \lambda_i v_i = v_0, \ \sum \lambda_i = 1\}$

UNBOUNDED $\{\max \ v_0^T y + y_0 \, ; \, v_i^T y \le -y_0 \ \forall k \in X\}$

INEQUALITIES

VALID $\pi^T x \le \pi_0$ FOR X IF $X \subseteq \{x \in \mathbb{R}^n : \pi^T x \le \pi_0\}$

REDUNDANT (1 TO 1) $\pi^T x \le \pi_0$ DOMINATES $\mu^T x \le \mu_0$ IF: $\cdot \ \pi \ge k\mu \ \cdot \ \pi_0 \le k$

(1-TO-MANY) $\pi^T x \le \pi_0$ RED FOR m INEQ IF: $\exists \bar{k} \in \mathbb{R}^m : \sum (k_i \mu_i)^T x \le \sum k_i \mu$

DOMINATES $\pi^T x \le \pi_0$

POLYTOPES & INEQUALIT. $DIM(P) = DIM(conv(X)) = AFF\_INDEP\_PTS - 1 = n - RANK(A^{=}_{m \times n})$

FACE 0 VERTEX 1 … n-1 FACE DIM(P)-1 FACET! :)

AFF. INDEP GIVEN $S = \{v_0, v_1, \dots v_m\} \Rightarrow \{v_1 - v_0, v_2 - v_0, \dots v_m - v_0\}$ LINEARL INDEP!

$\sum \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \ \forall i$

CHVATAL GOMORY

CG-INEQUALIT. ROUNDED LINEAR COMB. OF VALID INEQUALITIES $\lfloor \sum_i u_i A_i^T \rfloor x \le \lfloor \sum u_i b$

RANK FROM RANK (n-1) AND $Ax = b$ CHVATAL RANK $K : X^K = conv(\hat{P})$

GOMORY ALGORITHM D NON BASICS $a_{ij} = (B^{-1}A)_{ij} \quad b_i = (B^{-1}b)_i$

$$\sum_{j \in D} \underbrace{f_{ij}}_{\text{OF } A_{ij}} x_j \ge \underbrace{f_i}_{\text{OF } b_i}$$

DISJUNCTIVE INEQUALITIES

MODEL THE DISJOINT HULL OF X. $D=\{x\in\mathbb{R}^n : \bigvee_{k=1}^{q} D^k x \ge d_0^k\}$

VALID INEQUALITY ON X • $\sum_j \pi_j^i x_j \le \pi_0^i$ VALID FOR $i=1,2$ THEN

$\sum_j \min_j\{\pi_i^j\} x_j \le \max_j\{\pi_0^j\}$ IS VALID FOR X!

• $\pi^T x \le \pi_0$ VALID FOR CONV$(D_1 \cup D_2)$ IF

$\exists u_1,u_2\ge 0 : \pi^T \le u_i^T A_i \wedge \pi_0 \ge u_i^T b_i \quad i=1,2$

VALID INEQ. FOR IP POLYTOPES $S=\{P\cap \mathbb{N}^n\}$ $P_1=\{x\in\mathbb{R}^n_+ : \alpha^T x\le \alpha_0\}\cap P$ $P_2=\{x\in\mathbb{R}^n: \alpha^T x\ge \alpha_0+1\}$

WITH $\alpha,\alpha_0\in\mathbb{R}^n,\mathbb{R}$ => $\pi^T x\le\pi_0$ VALID FOR CONV$(P_1\cup P_2)$

1. $\pi^T \le u_1^T A + v_1\alpha^T$ 3. $\pi_0 \ge u_1^T b + v_1\alpha_0$

2. $\pi^T\le u_2^T A - v_2\alpha^T$ 4. $\pi_0\ge u_2^T b - v_2(\alpha_0+1)$) $\exists u_i\ge0, v_i\ge0$ $i=\{1,2\}$

CLIQUES, ODD CYCLES, COVERINGS

CLIQUES

SEPARATION PROBLEM $\max_s \sum_j (1-y_j^*) s_j$ S.T. $s_i+s_j\le1\ \forall i,j$ NOT IN CONFLICT

CONF. GRAPH V SET OF VAR $\bar{y}_j$ E (u,v) ARE IN CONFLICT

LIFTING ADD VAR IN CONFLICT WITH SOME IN CLIQUE MAX CARD = FACET

ODD-CYCLE IN THE FORM $\sum_j y_j \le 1/2(|J|-1)$ ALL IN CONFLICT

VIOLATED $(\sum_j y_j)\cdot 2 > |J|-1$

CONF. GRAPH V $(1-\bar y_u - \bar y_v)$ FOR NODES E (u,v) IN | J |

FIND AN ODD CYCLE IN G

LIFTING THROUGH SEP. PROBLEM. ASSUME C IN INEQUALITY +K

$z_K = \max_w \sum_{i\in C} d_i w_i$ S.T. $w_i+w_j\le1\ \forall(i,j)\in E(C\cup K)$

$w_K=1$ $w_i\in\mathbb{B}$ | $d_K = (|C|-1)/2 - z_K$ COEFF IN ODD-INEQ

KP01 COVER C: $\sum_{i\in C} x_i \le |C|-1$ $C\subseteq X : \sum_{i\in C} a_i > b$

MINIMAL TAKE OUT ANY ITEM => NOT A COVER

SEPARAT. PROBLEM $\min_w \sum_{i\in N}(1-\hat x_i) w_i \underline{<1}$ TO BE VALID $\hat x_i=0\ w_i=0$

S.T. $\sum_t a_t w_t \ge b+1$ $w\in\mathbb{B}^n$ $\hat x_i=1\ w_i=1$

LIFTING MINIMAL COVER $C=\{i_1,i_2,i_3\dots i_m\}$ $i_1<i_2<i_3\dots<i_m$

$\sum_{i<i_1}\alpha_i x_i + \sum_{i\in C} x_i \le |C|-1$ $\alpha_i = \begin{cases} 1 & \text{IF } a_{i_1}\le a_i\le a_{i_1}+a_{i_2} \\ 2 & \text{IF } a_{i_1}+a_{i_2}\le a_i \le a_{i_1}+a_{i_2}+a_{i_3} \\ r & \text{IF } \sum_{j=i_1}^{i_{r-1}} a_{i_j} \le a_i \le \sum_{j=i_1}^{i_r} a_{i_j} \end{cases}$

$r-1 \longrightarrow r$

DANTZIG WOLFE

SEPARATES COUPLING CONSTRAINTS FROM EASY ONES BLOCKANG.

MASTER PROBLEM $\min_\theta \sum_{p\in\Omega_i} c_p\theta_p$

$\sum_{p\in\Omega_i} a_p\theta_p = b$ (π)

$\sum_{p\in\Omega_i}\theta_p=1$ (u) ($u=\sigma$)

$\theta_p\ge0\in\mathbb{R}$ RELAX

OPTIMALITY IF $c_j\ge0\ \forall\theta_j\in\Omega\setminus\Omega_i$

SUB PROBLEM $c_p^i = c_p - \pi^i a_p - \sigma_i$

$\min \sum (c_i^T - (\pi_i)^T A_i) x - \sigma_i$

θ_i $Dx=e$ $x\in\mathbb{N}$

BRANCHING ON ORIGINAL VARIABLES! $x_k=\sum_{p\in\Omega}\theta_p w_p$

DOMAIN & BOUNDS $D_{DW}^{(INT)} \subseteq D_{DW}^{(CONT)} = D_{PL}$ IF SP IS NOT "IDEAL" => DW BETTER BOU

LAGRANGIAN RELAXATION

$z_P = \min c^Tx$

$A_1x=b_1$ π UNBOUNDED

$A_2x\ge b_2$ μ BOUNDED ≥0

$Dx=e$

=> $L(\pi,\mu)=\min_x c^Tx - \pi^T(A_1x-b_1)-\mu(A_2x-b_2)$ ST. $Dx=e$

// ARE A CONSTANT (OR SEPARABLE)

SEPARABLE $\Delta^K=\{x^K: Dx=e\}$ • PIECEW. LINEAR • CONCAVE

LOWER BOUND $L(\pi,\mu)\le z_P$

DUAL LAGR. $L^K=\{\max_{\mu,\pi} L(\pi,\mu)$ S.T. $\mu\ge0\}$ SUBGRAD: $J=-L(\pi,\mu)$ // $\partial f(\bar x)=\{\gamma\in\mathbb{R}^n: \forall x\in\mathbb{R}^n, f(x)\ge f(\bar x)+\gamma^T(x-\bar x)\}$

SUBGR. METHOD $(\pi_T,\mu_T)=[(\pi_{T-1},\mu_{T-1})-s_t\gamma_t(x^t)^T]^+$

+ SINCE $\mu\ge0$

ROBBINS-MONROE $\lim_{t\to\infty}s_t=0$ $\lim_{t\to\infty}\sum_t s_t=\infty$ OPTIMAL: $\gamma(x^*)=\begin{pmatrix}A_1x^*-b_1\\A_2x^*-b_2\end{pmatrix}$

$s_t \cong \alpha_t\in[0,2]\cdot(J(\pi_{t-1},\mu_{t-1})-\underline{z})\cdot\|\gamma_T\|^2$

BENDER DECOMPOS.

EASEIER SP RESOL WITH Y FIXED DUAL BLOCKANGULAR

MASTER PRIMAL $\min_{y\in Y} h(y)+d^Ty$ SUB PROB. $h(y)=\min_x c^Tx$ S.T. $Ax=e-By$ (u)

DUAL $u^T(e-By)$ S.T. $u^TA\le c$

FEASABILITY $u^T(e-By)\le0\ \forall u^T: u^TA\le0$ $u^T=\rho_r$ EXT. RAY

OPTIMALITY $z - w_p^T(e-By)\ge0\ \forall p\in\Omega$ $z=SUB\_P$ w_p = EXT. POIN

WARNING: MP MIGHT BE UNBOUNDED AT FIRST ITERATIONS!

SP: INF => FEAS CUT // UNB: UNB FEA OPT. CUT