A Koszul duality of partition Lie algebras(-oids)

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Let $X/\mathbb{C}$ be a smooth projective complex variety. The cohomology of tangent bundle $H^*(X, T_X)$ is naturally a graded Lie algebra over $\mathbb{C}$, and it controls the infinitesimal deformation of X. According to Kodaira–Spencer theory, $H^1(X, T_X)$ classifies the isomorphism classes of Cartesian diagrams as follows

$$\begin{array}{ccc} \mathscr{X} & \longleftarrow & X \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) & \longleftarrow & \mathrm{Spec}(\mathbb{C}) \end{array},$$

while $H^0(X, T_X)$ is isomorphic to the automorphism of trivial deformation $\mathscr{X}_0 = X \times \mathbb{C}$. Besides, for each $[\mathscr{X}] \in H^1(X, T_X)$, there is a class $c_{\mathscr{X}} \in H^2(X, T_X)$, such that $\mathscr{X}$ can be extended to $\mathbb{C}[\epsilon]/(\epsilon^3)$ if and only if $[c_{\mathscr{X}}, c_{\mathscr{X}}] = 0$.

It suggest that the derived global section $R\Gamma^*(X, T_X)[1]$ (with a shifting) should be regarded as a "groupoid" that classifies the infinitesimal deformations of X. More generally, a principle introduced by Deligne and Drinfeld postulates that every formal moduli problem in characteristic 0 is controlled by a dg-Lie algebra. Through the effort of many people, this principle was finally enhanced into a theorem by the work of Hinich [Hin01], Pridham [Pri10], and Lurie [Lur11].

Theorem 0.1. *Given a field $k/\mathbb{Q}$, there is an equivalence of ∞-categories*

$$\mathrm{MC} : \mathrm{Lie}_k^{\mathrm{dg}} \xrightarrow{\simeq} \mathcal{FMP}_k$$

identifying the homotopy theory of dg-Lie algebras and formal moduli problems.

Remark 0.2 (Recollection of $\mathcal{FMP}_k$)**.** A formal moduli problem X over k is a *good* functor (of ∞-categories)

$$X : \mathrm{Art}_k^* \to \mathcal{S},$$

where $A \in \mathrm{Art}_k^*$ is a connective cdga over k such that $\pi_0(A)$ is an ordinary augmented local Artinian ring and $\pi_*(A)$ is finitely generated over $\pi_0(A)$. A formal moduli problem X should satisfy that

- $X(k)$ is contractible;

- for each cospan $A_0 \xrightarrow{\rho} A_{01} \leftarrow A_1$ that $\pi_0(\rho)$ is surjective, then

$$\begin{array}{ccc} X(A_0 \times^h_{A_{01}} A_1) & \longrightarrow & X(A_1) \\ \downarrow & \lrcorner & \downarrow \\ X(A_0) & \xrightarrow[X(\rho)]{} & X(A_{01}) \end{array}$$

  is a homotopy pullback.

The thrust of proving Theorem 0.1 is an adjunction

$$\mathfrak{D} : \mathrm{cdga}_{k//k} \rightleftarrows (\mathrm{Lie}_k^{\mathrm{dg}})^{op} : C^*,$$

where $\mathfrak{D} : A \mapsto \mathbb{T}_{k/A}(\simeq \hom_k(\mathbb{L}_{k/A}, k))$. For each $\mathfrak{g} \in \mathrm{Lie}_k^{\mathrm{dg}}$, $C^*(\mathfrak{g})$ has a model whose underlying graded algebra is

$$\hom_k(\mathrm{Sym}_k(\mathfrak{g}[1]), k),$$

where the differential $d = d_1 + d_2$ is a combination of the internal differential d_1 of dual complex and d_2 defined by a formula form differential geometry: for each n-form $\omega \in (\mathrm{Sym}_k^n(\mathfrak{g}[1]))^\vee$ and $n+1$ vectors $X_0, \ldots, X_n \in \mathfrak{g}$,

$$d_2\omega(X_0, \ldots, X_n) = \sum \pm\omega([X_i, X_j], \ldots). \tag{1}$$

A dg-Lie algebra $\mathfrak{g}$ is said to be *coconnected* if $\pi_n(\mathfrak{g}) = 0$ for $n \geq 0$, and is said to be *of finite type* if every π_n is finitely dimensional. Denote $\mathrm{Lie}_{k,<0}^{\mathrm{dg,ft}} \subset \mathrm{Lie}_k^{\mathrm{dg}}$ the full subcategory of coconnected dg-Lie algebras of finite type.

Theorem 0.3 (Koszul duality of dg-Lie algebras)**.** *The functor C^* induces a fully faithful embedding*

$$C^* : \mathrm{Lie}_{k,<0}^{\mathrm{dg,ft}} \hookrightarrow (\mathrm{cdga}_{k//k})^{op},$$

whose essential image consists of connective completely Noetherian local algebras $A \to k$:

- $\pi_n(A) = 0$ *for* $n < 0$*;*
- $\pi_0(A)$ *is an I-complete Noetherian algebra, where I is the augmentation ideal of* $\pi_0(A) \to k$*;*
- $\pi_n(A)$ *is finitely generated over* $\pi_0(A)$*.*

Counterexample 0.4. The coconnectedness is **necessary** for Theorem 0.3. Set $k = \mathbb{R}$ the Lie algebra cohomology of $\mathfrak{su}(2)$ is the free cdga $C^*(\mathfrak{su}(2)) \simeq \wedge_{\mathbb{R}}(\mathbb{R}.e_3)$ with $|e_3| = -3$, cf.[MT91, Vol.I Theorem 6.5.(2)] [CE48, Theorem 15.2]. Therefore, the corresponding shifted dg-Lie algebra of $C^*(\mathfrak{su}(2))$ is abelian and has the underlying module $\mathbb{R}[3]$, which differs from $\mathfrak{su}(2)[1]$.

The dg-Lie algebras that are not coconnected (e.g. ordinary Lie algebras) are of great interest for formal geometry. It is natural to seek a Koszul duality for them using cdgas with more structures. Recall that $C^*(\mathfrak{g})$ has a model $\big((\mathrm{Sym}_k\,\mathfrak{g}[1])^\vee, d_1 + d_2\big)$, where $(\mathrm{Sym}_k\,\mathfrak{g}[1])^\vee$ has a second grading by the degree of forms, and d_1 preserve this grading, while d_2 increases it by 1, cf. (1). Thus, $C^*(\mathfrak{g})$ is naturally endowed with a complete multiplicative filtration $F^H C^*$ (called Hodge filtration) such that

$$F^H_n C^*(\mathfrak{g}) = \big((\mathrm{Sym}^{\geq n}_k\,\mathfrak{g}[1])^\vee, d_1 + d_2\big).$$

Toën–Vessozi conjectured that, when $\mathfrak{g}$ is finitely dimensional, the assignment $F^H C^*$ is a fully faithful embedding into augmented filtered cdgas[1]. Additionally, Brantner–Mathew generalised Lurie–Pridham's theorem into positive characteristics using *partition Lie algebras*. Therefore, it might be interesting to investigate the following question:

Target

Establishing a Koszul duality for non-coconnective Lie algebras in arbitrary characteristics, utilizing the Hodge filtration.

The first challenge is choosing a suitable algebraic context: cdgas and dg-Lie algebras behave badly in positive characteristics as quasi-isomorphisms and the expected fibrations do not induce a model structure.

Example 0.5. Consider a morphism of cdgas $f : A \to B$ over $\mathbb{F}_p$. There is no factorization of f into a trivial cofibration followed by a fibration (degrewise surjection). Otherwise, there is $A \xrightarrow{g,\simeq} A' \xrightarrow{p} B$, where, for each $x \in pi_*(B)$, there is some $y \in \pi_*(A') \cong \pi_*(A)$ sucht that $p(y) = x^p$. However, such a lifting does not exist in general.

Away from characteristic 0, there are two inequivalent natural generalizations of commutative rings, that are $\mathbb{E}_\infty$-ring spectra and animated (commutative) rings. We choose animated rings[2] for our purpose in derived algebraic geometry. Recall that the ∞-category of animated rings is defined as

$$\mathrm{AniRing} := \mathrm{Fun}_\Sigma(\mathrm{Poly}^{op}, \mathcal{S}),$$

where Σ means taking the full subcategory of sifted-colimit-preserving functors. There is a monadic free-forgetful adjunction

$$\mathrm{Mod}_{\mathbb{Z},\geq 0} \rightleftarrows \mathrm{AniRing}$$

whose monad gives rise to the left derived functor LSym of ordinary Sym.

We have seen that the Koszul duality of non-coconnective "Lie algebras" should be some non-connective filtered algebras. A non-connective generalistion

[1] Their conjecture is more general, which is formulated for *dg-Lie algebroids*, cf. [TV23, §1.3.1] for the question and [Fu24, Main Theorem 1.2] for an answer

[2] We say simplicial commutative rings in Europe (at least in France).

of AniRing requires to extend LSym into a sifted-colimit-preserving functor acting on Sp. The crucial method is the right-left extension in [BM19, §3][BCN21, §2]. We also recollect the basics of (generalized) ∞-operads, which helps to treat different types algebras uniformly.

1 Right-left derived functors

An *additive ∞-category* $\mathscr{A}$ is an ∞-category with finite products and coproducts such that $h\mathscr{A}$ is an ordinary additive category. The ∞-category of left $\mathscr{A}$-modules is defined as

$$\mathrm{Mod}_{\mathscr{A}} := \mathrm{Fun}_{\oplus}(\mathscr{A}^{op}, \mathrm{Sp}),$$

where $\oplus$ means direct-sum-preserving.

Example 1.1. Let R be an ordinary ring and $\mathscr{A} := \mathrm{Vect}_R^{\omega}$ be the category of finitely generated free modules. The module category $\mathrm{Mod}_{\mathscr{A}}$ is precisely Mod_R, the unbounded derived ∞-category of R-chain complexes.

A left $\mathscr{A}$-module is said to be perfect if it is compact in $\mathrm{Mod}_{\mathscr{A}}$. The full subcategory Perf_R of perfect left $\mathscr{A}$-modules is the minimal stable sub-∞-category containing $\mathscr{A}$ and closed under retractions. Moreover, the natural pairing

$$\mathrm{Perf}_{\mathscr{A}} \times \mathrm{Perf}_{\mathscr{A}^{op}} \to \mathrm{Sp}$$

defined by extending $(A, B) \mapsto \hom(B, A)$ gives rise to an equivalence $\mathrm{Perf}_{\mathscr{A}} \simeq \mathrm{Perf}_{\mathscr{A}^{op}}^{op}$. Denote the essential image of $\mathrm{Perf}_{\mathscr{A}^{op},\geq 0}$ in $\mathrm{Perf}_{\mathscr{A}}$ as $\mathrm{Perf}_{\mathscr{A},\leqslant 0}$, which consists of the *dually connective perfect modules.*

Proposition 1.2. *Let $\mathscr{A}$ be an additive ∞-category and $\mathcal{V}$ an ∞-category with sifted colimits. Then, the restriction*

$$\mathrm{Fun}_{\Sigma}(\mathrm{Mod}_{\mathscr{A}}, \mathcal{V}) \xrightarrow{\simeq} \mathrm{Fun}_{\sigma}(\mathrm{Perf}_{\mathscr{A},\leqslant 0}, \mathcal{V})$$

is an equivalence, whose inverse is given by left Kan extensions. Here, σ means preserving finite stable geometric realizations.

Proof. Cf. [BCN21, Propsition 2.40]. □

In particular, left Kan extension induces a monoidal equivalence

$$\mathrm{End}_{\Sigma}(\mathrm{Perf}_{\mathscr{A},\leqslant 0}) \xrightarrow{\simeq} \mathrm{End}_{\Sigma}^{\mathrm{Perf}_{\mathscr{A},\leqslant 0}}(\mathrm{Mod}_{\mathscr{A}}),$$

where the right-hand side consists of sifted-colimit-preserving endo-functor that preserves $\mathrm{Perf}_{\mathscr{A},\leqslant 0}$.

Brantner–Campos–Nuiten also provides a practical method to obtain functors in $\mathrm{Fun}_{\sigma}(\mathrm{Perf}_{\mathscr{A},\leqslant 0}, \mathcal{V})$, cf. [BCN21, Proposition 2.46]:

Proposition 1.3. *Let $\mathscr{A}$ be an additive ∞-category and $\mathcal{V}$ an ∞-category with sifted colimits. If $F : \mathscr{A} \to \mathcal{V}$ is the colimit of a coutable sequece*

$$F_1 \to F_2 \to \dots,$$

where each $F_i : \mathscr{A} \to \mathcal{V}$ is of finite degree. Then, the right Kan extension F^R of F along $\mathscr{A} \hookrightarrow \mathrm{Perf}_{\mathscr{A},\leqslant 0}$ belongs to $\mathrm{Fun}_\sigma(\mathrm{Perf}_{\mathscr{A},\leqslant 0}, \mathcal{V})$.

Given such an $F : \mathscr{A} \to \mathcal{V}$, the left extension F^{RL} of F^R is called the *right-left derived functor of F*, which is sifted-colimit-preserving.

Since the ordinary symmetric power Sym admits a splitting filtration $\mathrm{Sym} = \oplus_{n\in\mathbb{N}} \mathrm{Sym}^n$, it suits into the case of Proposition 1.3. We obtain a right-left derived functor LSym of Sym acting on $\mathrm{Mod}_\mathbb{Z}$ as expected. However, as LSym does not preserve perfect modules, the above propositions give cannot give a monad structure on LSym.

This problem is overcame by considering *derived ∞-operads*.

We start with underived symmetric sequences. Let $B\Sigma$ be the ordinary 1-category of finite sets and isomorphisms, which supports naturally a cocartesian symmetric monoidal structure. For any presentalbe symmetric monoidal ∞-category $\mathcal{C}$, its ∞-category of symmetric sequences

$$\mathrm{sSeq}(\mathcal{C}) := \mathrm{Fun}(N(B\Sigma), \mathcal{C})$$

admits a natural symmetric monoidal structure $\otimes$ given by Day convolution. Moreover, $\mathrm{sSeq}(\mathcal{C})$ has the universal property of being a symmmetric monoidal ∞-category under $\mathcal{C}$ freely generated by $\mathbb{1}$, posing a unit of $\mathcal{C}$ at arity 1, i.e.

$$F \mapsto F(\mathbb{1}) : \mathrm{End}^L_{\mathcal{C}/}(\mathrm{sSeq}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{sSeq}(\mathcal{C}).$$

The *composite product* $\circ$ is the opposite of the composition in $\mathrm{End}^L_{\mathcal{C}/}(\mathrm{sSeq}(\mathcal{C}))$ and satisfies the formula

$$M \circ N \simeq \oplus_{n\in\mathbb{N}}(M(n) \otimes N^{\otimes n})_{h\Sigma_n},$$

which agrees to the picture of decorated tree grafting. An ∞-operad refers to an $\mathbb{A}_\infty$-algebra in $\mathrm{sSeq}(\mathcal{C})$. For instance, the unit of levelwise tensor product admits a natural operadic structure, which governs the $\mathbb{E}_\infty$-algebras in $\mathcal{C}$.

When $\mathcal{C} = \mathrm{Mod}_R$ for some ordinary commutative ring R, we want to consider a derived variant of $\mathrm{sSeq}(\mathcal{C})$. The motivation is simple: as homotopy orbit $(-)_{h\Sigma_n}$ does not send $\mathrm{Mod}^\heartsuit_{R[\Sigma_n]}$ to discrete R-modules for $n \geq 2$ and general R, $\mathrm{sSeq}(\mathrm{Mod}_R)$ does not have enought objects to encode LSym. To fix this, we should include the finite free R-modules with non-free Σ_n-action as projective generators. More precisely, consider the smallest additive sub-1-category

$$R[\mathcal{O}_{\Sigma_n}] \subset \mathrm{Mod}^\heartsuit_{R[\Sigma_n]}$$

containing $R[\Sigma_n/H]$, the equivariant R-modules generated by some $H < \Sigma_n$, and write $\oplus_{n\in\mathbb{N}} R[\mathcal{O}_{\Sigma_n}] \subset \mathrm{sSeq}(\mathrm{Mod}_R)$ as $R[\mathcal{O}_\Sigma]$[3].

[3]Please keep in mind that this is a formal notation rather an actual group ring.

Definition 1.4. The ∞-category of *derived symmetric sequences* or *genuine symmetric sequences* over R is defined as

$$\mathrm{sSeq}_R^{\mathrm{gen}} := \mathrm{Mod}_{R[\mathcal{O}_\Sigma]} .$$

The next step is to construct the derived analogues of $\otimes$, $\otimes_{lev}$ and $\circ$ on $\mathrm{sSeq}_R^{\mathrm{gen}}$. One can observe that $R[\mathcal{O}_\Sigma]$ is closed under the truncated monoidal structures $\otimes$, $\otimes_{lev}$ and $\circ$ in $\mathrm{sSeq}_R^{\heartsuit}$. Besides, these monoidal structures are *locally polynomial* in the following sense:

Definition 1.5. A functor $F : \mathscr{A} \to \mathscr{B}$ between additive ∞-categories is said to be *locally polynomial* if (1) F is a countable sequential colimit $(F_1 \to F_2 \to \dots)$ of $F_i : \mathscr{A} \to \mathscr{B}$ functors of finite degree and (2) the sequence $F_1(X) \to F_2(X) \to \dots$ stabilizes at some point for every $X \in \mathscr{A}$.

Theorem 1.6. *Let $\mathcal{A}dd^{\mathrm{poly}}$ be the ∞-category of additive ∞-category and locally polynomial functors. Then, there is a functor*

$$\mathrm{Mod}_{(-)} : \mathcal{A}dd^{\mathrm{poly}} \to \mathcal{P}r^{\mathrm{st},\Sigma}.$$

Proof. Cf. [BCN21, Theorem 2.52] □

Therefore, the right-left derived functors of $\otimes$, $\otimes_{lev}$ and $\circ$ induce monoidal structures on $\mathrm{sSeq}_R^{\mathrm{gen}}$ for ordinary commutative ring R, where we keep the same notation. More generally, $R[\mathcal{O}_\Sigma]$ and $\mathrm{sSeq}_R^{\mathrm{gen}}$ can be defined for *arbitrary* animated ring R using spectral Mackey functors, cf. [BCN21, §2.1, §3.5]. The monoidal structures on $\mathrm{sSeq}_R^{\mathrm{gen}}$ can be deduced from the fact that

$$(R \mapsto R[\mathcal{O}_\Sigma]) : \mathrm{AniRing} \to \mathcal{A}dd$$

is sifted-colimit-preserving, cf. [Fu24, Lemma 2.50].

Definition 1.7. Let R be an animated ring.

(1) The ∞-category of *derived ∞-operads* is defined as $\mathrm{Alg}(\mathrm{sSeq}_R^{\mathrm{gen}}, \circ)$.

(2) The module category Mod_R can be regarded as a left categorical ideal of $(\mathrm{sSeq}^{\mathrm{gen}}, \circ)$ by embedding into arity 0. Then, for some derived ∞-operad $\mathcal{P}$, a $\mathcal{P}$-algebra is by definition a left $\mathcal{P}$-module in Mod_R.

The unit of levelwise tensor product admits a natural derived operadic structure, denoted as Com, whose algebra category $\mathrm{DAlg}(\mathrm{Mod}_R)$ agrees with the ∞-category DAlg_R of derived rings over R introduced in [Rak20, §4]. In particular, $\mathrm{AniRing}_R$ can be ragarded as the full subcategory of $\mathrm{DAlg}(\mathrm{Mod}_R)$ spanned by connective algebras.

Now, we close this section by a recollection of filtered algebras. Consider the ∞-category of filtered R-modules

$$\mathrm{Fil}\,\mathrm{Mod}_R := \mathrm{Fun}(N(\mathbb{Z}^{\leq}), \mathrm{Mod}_R),$$

where the objects can be written as $(\ldots \to F_1X \to F_0X \to F_{-1}X \to \ldots)$. The ∞-category $\mathrm{fil}\,\mathrm{Vect}_R^\omega$ of finite free R-modules with a splitting filtration on the basis is an additive sub-∞-category that generates $\mathrm{Fil}\,\mathrm{Mod}_R$, i.e.

$$\mathrm{Mod}_{\mathrm{fil}\,\mathrm{Vect}_R^\omega} \xrightarrow{\simeq} \mathrm{Fil}\,\mathrm{Mod}_R\,.$$

Then, there is a filtered notion of derived symmetric sequences

$$\mathrm{sSeq}_{R,\mathrm{Fil}}^{\mathrm{gen}}$$

equipped with the monoidal structures $\otimes$, $\otimes_{lev}$ and $\circ$.

The symmetric monoidal embedding to filtration degree 0

$$\big(M \mapsto (\ldots 0 \to M \to M \to \ldots)\big) : \mathrm{Mod}_R \hookrightarrow \mathrm{Fil}\,\mathrm{Mod}_R$$

induces a fully faithful functor

$$(-)_0 : \mathrm{sSeq}_R^{\mathrm{gen}} \hookrightarrow \mathrm{sSeq}_{R,\mathrm{Fil}}^{\mathrm{gen}}\,.$$

Particularly, there is a left $\mathrm{sSeq}_R^{\mathrm{gen}}$-tensored structure on $\mathrm{Fil}\,\mathrm{Mod}_R$.

Let $\mathrm{sSeq}_R^{\mathrm{gen,red}} \subset (\mathrm{sSeq}_R^{\mathrm{gen}})_{\mathbb{1}//\mathbb{1}}$ be the full subcategory of *reduced* objects M, which means that $M(0) = 0$ and the chosen map $\mathbb{1} \to M(1)$ is an equivalence. The functor $(-)_0$ induces a left $\mathrm{sSeq}_R^{\mathrm{gen,red}}$-tensored structure on

$$\mathrm{Fil}_{\geq 1}\,\mathrm{Mod}_R \subset \mathrm{Fil}\,\mathrm{Mod}_R$$

spanned by such $F_\bullet M$ that stabilizes when degree ≥ 1, i.e.

$$\ldots \to F_3M \to F_2M \to F_1M \xrightarrow{id} F_1M \xrightarrow{id} F_1M \to \ldots,$$

and similarly a left $\mathrm{sSeq}_R^{\mathrm{gen,red}}$-tensored structure on $\mathrm{Fil}_{\leq -1}\,\mathrm{Mod}_R \subset \mathrm{Fil}\,\mathrm{Mod}_R$ spanned by $F_\bullet M$ such that $F_nM \simeq 0$ for $n \geq 0$. The same method also produces natural left $\mathrm{sSeq}_R^{\mathrm{gen,red}}$-actions on the graded module categories $\mathrm{Gr}_{\geq 1}\,\mathrm{Mod}_R$ and $\mathrm{Gr}_{\leq -1}\,\mathrm{Mod}_R$.

The derived ∞-operad Com^{nu} of *non-unital derived algebras* has arity ≥ 1 components the same as Com but $\mathrm{Com}^{nu}(0) = 0$. There are adjunctions of non-unital derived algebras

$$\mathrm{DAlg}_R^{nu} \underset{F^1}{\overset{\mathrm{adic}}{\rightleftarrows}} \mathrm{DAlg}^{nu}(\mathrm{Fil}_{\geq 1}\,\mathrm{Mod}_R) \overset{\mathrm{Gr}}{\rightleftarrows} \mathrm{DAlg}^{nu}(\mathrm{Gr}_{\geq 1}\,\mathrm{Mod}_R)\ .$$

2 Divided power Koszul duality

Given some $R \in \mathrm{AniRing}$, there is an adjunction

$$\mathrm{cot} : \mathrm{DAlg}_R^{nu} \rightleftarrows \mathrm{Mod}_R : \mathrm{sqz}$$

where $\cot(A) = \mathbb{L}_{R/A}[-1]$ with A identified with the corresponding augmented R-algebra, and $\mathrm{sqz}(M)$ equipps M with the trivial algebra structure. Then, there is a natural monad

$$T^{naïve} := (\cot \circ \mathrm{sqz}(-)^{\vee})^{\vee}$$

that might be useful for defining the *derived partition Lie algebras*. Unfortunately, this monad is not satisfying as it does not preserve sifted colimits.

Brantner–Mathew rectified this functor by finding a comonadic restriction of $\cot \dashv \mathrm{sqz}$, cf. [BM19, Theorem 4.20]. We adopt the approach of PD Koszul duality of operads introduced in [BCN21]. It consists of roughly two step: (1) the functor of cotangent fibre cot could be recovered by bar-cobar adjunction between derived ∞-operads and cooperads; (2) taking R-linear dual sends derived ∞-cooperads to "derived ∞-operads" with divided powers.

The core of the first step is a categorical bar-cobar construction established in [Lur17, §5.2.2] and refined in [BCN21, §3.4].

Theorem 2.1. *Let $\mathcal{C}$ be a pointed monoidal ∞-category, and $\mathcal{M}$ be a left $\mathcal{C}$-tensored ∞-category. If both $\mathcal{C}$ and $\mathcal{M}$ admit geometric realizations, there is a commuting diagram*

$$\begin{array}{ccc} \mathrm{LMod}(\mathcal{M}) & \underset{\mathrm{coBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & \mathrm{LComod}(\mathcal{M}) \\ \pi\downarrow & & \downarrow\pi \\ \mathrm{Alg}(\mathcal{C}) & \underset{\mathrm{coBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathrm{Alg}(\mathcal{C}) \end{array},$$

where the horizontal arrows are adjunctions.

Here, the ∞-category $\mathrm{LMod}(\mathcal{M})$ consists of pairs (A, M), where $A \in \mathrm{Alg}(\mathcal{C})$ and M is a left A-module in $\mathcal{M}$. The ∞-category $\mathrm{LComod}(\mathcal{M})$ is spanned by pairs (C, N) of coalgebras and comodules. For each $A \in \mathrm{Alg}(\mathcal{C})$, $\mathrm{Bar}(A)$ is calculated by $1 \otimes_A 1$, whose comultiplication is $1 \otimes_A 1 \simeq 1 \otimes_A A \otimes_A 1 \to 1 \otimes_A 1 \otimes_A 1$. Similarly, for each A-module M, $\mathrm{Bar}(M) \simeq 1 \otimes_A M$ is equipped with a natural $\mathrm{Bar}(A)$-comodule structure.

Taking $\mathcal{C} = \mathrm{sSeq}_R^{\mathrm{gen,red}}$, there is an ajunction

$$\mathrm{Bar} : \mathrm{Op}_R^{\mathrm{gen,red}} \rightleftarrows co\,\mathrm{Op}^{\mathrm{gen,red}} : \mathrm{coBar}$$

between reduced derived ∞-operads and reduced derived ∞-cooperads. Brantner–Heuts claim that the unit map $id \to \mathrm{coBar} \circ \mathrm{Bar}$ is an equivalence in a forthcoming project, using Ching's strategy (cf. [Chi12]) for spectral ∞-operads.

Definition 2.2. The *coLie derived ∞-cooperad* is defined as

$$co\,\mathrm{Lie}^{\pi}_{R,\Delta} := \mathrm{Bar}(\mathrm{Com}^{nu}).$$

The subscript Δ means that this is defined in the derived context in contrast to spectral context, while π hints the divided powers that do not show up yet.

The next step is to define *derived partition Lie algebras* by considering the R-linear dual of $co\,\mathrm{Lie}^{\pi}_{R,\Delta}$. However, the underlying module of $co\,\mathrm{Lie}^{\pi}_{R,\Delta}$ is not perfect. In fact, the underlying (non-derived) symmetric sequence of $co\,\mathrm{Lie}^{\pi}_{R,\Delta}$ is the same as usual shifted coLie cooperad over R, which means that $co\,\mathrm{Lie}^{\pi}_{R,\Delta}(p) = \Sigma^{p-1}R$ for $p \geq 1$. Thus, there is no good notion of duality for $co\,\mathrm{Lie}^{\pi}_{R,\Delta}$-coalgebras. Nevertheless, $co\,\mathrm{Lie}^{\pi}_{R,\Delta}$ is *almost perfect* in $\mathrm{sSeq}^{\mathrm{gen}}_R$. We can still obtain a meaningful duality by switching to the context of *pro-coherent modules* for suitable R.

Definition 2.3. Given $\mathscr{A}$ an additive ∞-category, a left A-module M is said to be *almost perfect* if, for every $n \in \mathbb{N}$, there exist some morphism $f_n : P_n \to M$ from a perfect module P_n such that $\mathrm{fib}(f_n)$ is n-connective.

Let $\mathrm{Aperf}_{\mathscr{A}} \subset \mathrm{Mod}_{\mathscr{A}}$ denote the full subcategory of almost perfect modules.

Intuitively speaking, an almost perfect module M is a bounded below module with finite generators on each degree. More precisely, $\mathrm{Aperf}_{\mathscr{A}}$ is the smallest stable sub-∞-category of $\mathrm{Mod}_{\mathscr{A}}$ that contains $\mathscr{A}$ and closed under geometric realizations, cf. [Lur17, Proposition 7.2.4.11] for details.

There is a natural t-structure on $\mathrm{Mod}_{\mathscr{A}} = \mathrm{Fun}(\mathscr{A}^{op}, \mathrm{Sp})$ transported from Sp. An additive ∞-category $\mathscr{A}$ is said to be *left coherent* if $\mathrm{Aperf}_{\mathscr{A}}$ inherits a t-structure from that on $\mathrm{Mod}_{\mathscr{A}}$, *coherent* if both $\mathscr{A}$ and $\mathscr{A}^{op}$ are left coherent.

Definition 2.4. For a coherent $\mathscr{A}$, the ∞-category of *pro-coherent* left $\mathscr{A}$-modules is defined as

$$\mathrm{QC}^{\vee}_{\mathscr{A}} := \mathrm{Fun}_{ex,conv}(\mathrm{Aperf}_{\mathscr{A}^{op}}, \mathrm{Sp})$$

the ∞-category of exact and *convergent*, where a functor $F : \mathrm{Aperf}_{\mathscr{A}} \to \mathrm{Sp}$ is said to be *convergent* if, for every $X \in \mathrm{Aperf}_{\mathscr{A}^{op}}$, the natural morphism $F(X) \to \lim_n F(\tau_{\leq n} X)$ is an equivalence.

The Yoneda functor induces a fully faithful embedding

$$\mathrm{Aperf}_{\mathscr{A}^{op}} \hookrightarrow \mathrm{QC}^{\vee}_{\mathscr{A}},$$

whose essential image $\mathrm{Aperf}^{\vee}_{\mathscr{A}}$ is called as *dually almost perfect left $\mathscr{A}$-modules.*

Example 2.5. Let R be a coherent animated ring. Then, the additive ∞-categories Vect^{ω}_R, $\mathrm{fil}\,\mathrm{Vect}_R$, $\mathrm{gr}\,\mathrm{Vect}^{\omega}_R$ and $R[\mathcal{O}_\Sigma]$ are all coherent. It gives rise to the ∞-categoriesof (filtered, graded) pro-coherent R-modules $\mathrm{QC}^{\vee}_R$, $\mathrm{Fil}\,\mathrm{QC}^{\vee}_R$ and $\mathrm{Gr}\,\mathrm{QC}^{\vee}_R$, and also the ∞-category of *pro-coherent derived symmetric sequences* $\mathrm{sSeq}^{\mathrm{gen},\vee}_R$.

Additionally, these additive ∞-categories are closed under R-linear dual (with respect to $\otimes_{lev}$ for $R[\mathcal{O}_\Sigma]$), which induces contravariant autoequivalences.

The theory of right-left functors applies to pro-coherent modules as well.

Theorem 2.6 (Pro-coherent refinement of Theorem 1.6)**.** *There is a natural transformation of symmetric monoidal functors*

$$\iota : \mathrm{Mod} \Rightarrow \mathrm{QC}^{\vee} : \mathcal{A}dd^{\mathrm{coh,poly}} \to \mathcal{P}r^{\mathrm{st},\Sigma}$$

sending each locally polynomial functor to its right-left derived functor.

Proof. Cf. [BCN21, Theorem 2.52]. □

Particularly, if $\mathscr{A}$ has a locally polynomial (symmetric) monoidal structure, then the right-left functor exhibits on $\mathrm{QC}^{\vee}_{\mathscr{A}}$ a sifted-colimit-preserving (symmetric) monoidal structure. In the framework of pro-coherent modules, (dually) almost perfect modules enjoy a good duality:

Proposition 2.7. *Let $\mathscr{A}$ be a coherent additive ∞-category endowed wih a non-unital symmetric monoidal structure $\otimes$ preserving finite direct sums. Assume that the induced $\otimes$ on $\mathrm{QC}^{\vee}_{\mathscr{A}}$ admits an eventually connective unit $\mathbb{1}$, and each object in $\mathscr{A}$ is dualizable with the dual object existing in $\mathscr{A}$. Then taking duals gives rise to an equivalence,*

$$(-)^{\vee} := \hom_{\mathrm{QC}^{\vee}_{\mathscr{A}}}(-, \mathbb{1}) : \mathrm{Aperf}_{\mathscr{A}} \xrightarrow{\simeq} \mathrm{Aperf}^{\vee, op}_{\mathscr{A}} .$$

Proof. Cf. [BCN21, Proposition 2.55]. □

Powered by this duality, we are about to reach a statement like

The almost perfect coalgebras of some almost perfect derived ∞-cooperd $\mathcal{Q}$ are equivalent to the dually almost perfect algebras of $\mathcal{Q}^{\vee}$.

However, we should be careful about what kind of monad structure is put on $\mathcal{Q}^{\vee}$. Recall that the composite product $\circ$ $\mathrm{sSeq}^{\mathrm{gen},\vee}_{R}$ is defined by

$$M \circ N \simeq \oplus_{n \in \mathbb{N}} (M(n) \otimes N^{\otimes n})_{\Sigma_n},$$

where $(-)_{\Sigma_n}$ refers to the right-left derived functor of ordinary orbit. At the same time, its R-linear dual is

$$(M \circ N)^{\vee} \simeq \prod_{n \in \mathbb{N}} \big((M(n) \otimes N^{\otimes n})^{\vee} \big)^{\Sigma_n},$$

where $(-)^{\Sigma_n}$ referst to the derived functor of ordinary fixed points. It means that $(-)^{\vee}$ is not lax monoidal with respect to $\circ$. Instead, we should consider the *restricted product* $\bar{\circ}$ satisfying the formula

$$M \,\bar{\circ}\, N \simeq \oplus_{n \in \mathbb{N}} (M(n) \otimes N^{\otimes n})^{\Sigma_n}.$$

The restricted product is the origin of divided powers.

Definition 2.8. Let R be a coherent animated ring.

(1) A *derived PD ∞-operad* over R refers to an $\mathbb{A}_\infty$-algebra in $(\mathrm{sSeq}_R^{\mathrm{gen},\vee}, \bar{\circ})$.

(2) The *PD Koszul dual* of a reduced derived ∞-operad $\mathcal{P}$ is defined as $\mathrm{KD}^{\mathrm{pd}}(\mathcal{P}) := \mathrm{Bar}(\mathcal{P})^\vee$ with the natural derived PD ∞-operad structure. More specially, $\mathrm{Lie}^{\pi}_{R,\Delta} := \mathrm{KD}^{\mathrm{pd}}(\mathrm{Com}^{nu})$ is called the *derived partition Lie operad*.

Roughly speaking, a *(derived) partition Lie algebra* L over R is a pro-coherent R module equpped with a left $\mathrm{Lie}^{\pi}_{R,\Delta}$-action

$$\mu : \oplus_{n\geq 1}(\mathrm{Lie}^{\pi}_{R,\Delta}(n) \otimes L^{\otimes n})^{\Sigma_n} \to L.$$

3 A filtered Koszul duality of $\mathrm{Lie}^{\pi}_{R,\Delta}$-algebras

The functoriality of Theorem 2.6 and the laxity of

$$(-)^\vee : \left(\mathrm{sSeq}_R^{\mathrm{gen},\vee,op}, \circ\right) \to \left(\mathrm{sSeq}_R^{\mathrm{gen},\vee}, \bar{\circ}\right)$$

induces a commuting diagram of adjunctions as follows

$$\begin{array}{ccccc}
\mathrm{DAlg}^{nu}(\mathrm{QC}_R^\vee) & \underset{\mathrm{coBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathrm{Alg}_{co\,\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{QC}_R^\vee) & \underset{(-)^{\mathsf{i}}}{\overset{(-)^\vee}{\rightleftarrows}} & \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{QC}_R^\vee)^{op} \\
\mathrm{adic}\downarrow\uparrow F^1 & & (-)_1\downarrow\uparrow F^1 & & \mathrm{const}\downarrow\uparrow \mathrm{colim} \\
\mathrm{DAlg}^{nu}(\mathrm{Fil}_{\geq 1}\mathrm{QC}_R^\vee) & \underset{\mathrm{coBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathrm{Alg}_{co\,\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{Fil}_{\geq 1}\mathrm{QC}_R^\vee) & \underset{(-)^{\mathsf{i}}}{\overset{(-)^\vee}{\rightleftarrows}} & \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{Fil}_{\leq -1}\mathrm{QC}_R^\vee)^{op} \\
\mathrm{Gr}\downarrow\uparrow & & \mathrm{Gr}\downarrow\uparrow & & \mathrm{Gr}\downarrow\uparrow \\
\mathrm{DAlg}^{nu}(\mathrm{Gr}_{\geq 1}\mathrm{QC}_R^\vee) & \underset{\mathrm{coBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathrm{Alg}_{co\,\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{Gr}_{\geq 1}\mathrm{QC}_R^\vee) & \underset{(-)^{\mathsf{i}}}{\overset{(-)^\vee}{\rightleftarrows}} & \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{Gr}_{\leq -1}\mathrm{QC}_R^\vee)^{op}
\end{array}.$$

Here, $(-)^{\mathsf{i}}$ is formally given by the adjoint functor theorem, whose underlying module can be calculated as $(L)^\vee$ for dually almost perfect L. The functor const send partition Lie algebras L to $(L \xrightarrow{id} L \to \dots)$ the constant filtered algebras, where $\mathrm{colim}(F_* L)$ is taking the underlying object.

Definition 3.1. The *Hodge-filtered Chevalley–Eilenberg complex* $\widetilde{C}^*$ is defined as the composition

$$\left(\widetilde{C}^* := \mathrm{coBar} \circ (-)^{\mathsf{i}} \circ \mathrm{colim}\right) : \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{QC}_R^\vee) \to \mathrm{DAlg}^{nu}(\mathrm{Fil}_{\geq 1}\mathrm{QC}_R^\vee)^{op}.$$

The filtration on $\widetilde{C}^*$ is always complete by construction, and this functor sends sifted colimits to sifted limits. When R is a regular ring over $\mathbb{Q}$, $\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{QC}_R^\vee)$ agrees with the ∞-category of (shifted) dg-Lie algebras over R, and $\widetilde{C}^*$ can be modeled by the explicit filtration defined before.

Theorem 3.2. *There is a fully faithful embedding*

$$\widetilde{C}^* : \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{R,\Delta}}(\mathrm{Aperf}_R^\vee) \hookrightarrow \mathrm{DAlg}^{nu}(\mathrm{Fil}_{\geq 1}\mathrm{QC}_R^\vee)^{op},$$

whose essential image consists of complete $A \to R$ such that $\mathrm{Gr}_1 A$ is almost perfect over R, and the natural morphism $\mathrm{LSym}_R(\mathrm{Gr}_1)A \to \mathrm{Gr}\, A$ of graded algebras is an equivalence.

Sketch of proof. (Cf. [Fu24, Theorem 3.25]) The functor const is obviously fully faithful, so we can consider the filtered $\mathrm{Lie}^{\pi}_{R,\Delta}$-algebras in the form of

$$(L \xrightarrow{id} L \to \dots)$$

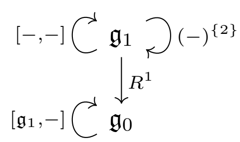
with L dually almost perfect. Its dual is a complete filtered $co\,\mathrm{Lie}^{\pi}_{R,\Delta}$-colagebra

$$\dots \to 0 \to L^{\vee},$$

whose graded pieces form a trivial coalgebra. Then, we show that coBar sends a trivial graded $co\,\mathrm{Lie}^{\pi}_{R,\Delta}$-coalgebra C to the free graded derived algebra $\mathrm{LSym}_R\, C$. By our construction, Gr commutes with everything, so $\mathrm{Gr}\, \widetilde{C}^*(L) \simeq \mathrm{LSym}_R\, L^{\vee}$ as graded drived algebras.

The above calculation shows that $\mathrm{KD}^{\mathrm{pd}} \circ \widetilde{C}^*(L)$ is a constant filtered $\mathrm{Lie}^{\pi}_{R,\Delta}$-algebra whose underlying algebra is L itself. $\square$

Example 3.3. Set $k = \mathbb{F}_2$ the field with two elements. The homotopy operations on a $\mathrm{Lie}^{\pi}_{k,\Delta}$-algebra L concentrating in degree 0 and 1 consist of $([-,-], (-)^{\{2\}})$ a restricted Lie structure on $\mathfrak{g}_1 := \pi_1(L)$, a $\mathfrak{g}_1$-representation structure on $\mathfrak{g}_0 := \pi_0(L)$ and a new additive operation $R^1 : \mathfrak{g}_1 \to \mathfrak{g}_0$.



For instance, the Frobenius kernel $\mu_2 := \ker(\mathbb{G}_{m,k} \xrightarrow{(-)^2} \mathbb{G}_{m,k})$ is an infinitesimal group with the underlying scheme $\mathrm{Spec}(k[x]/(x^2))$. The formal moduli problem $B\mu_2$ correspond to a $\mathrm{Lie}^{\pi}_{k,\Delta}$-algebra $L \simeq k.D_1 \oplus k.D_0$ with $|D_i| = i$. Its homotopy operations are determined by $[D_i, D_j] = 0$, $(D_1)^{\{2\}} = D_1$ and $R^1(D_1) = D_0$, cf. [Fu24, §3.4] for details.

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